BACKGROUND IN SYMPLECTIC GEOMETRY

NILAY KUMAR

Today I want to introduce some of the symplectic structure underlying classical mechanics.

The key idea is actually quite old and in its various formulations it dates back to the seventeenth to eighteenth centuries. This is the variational approach to mechanics:

> **Meta-theorem:** the path taken by a system between two points in time is the one for which a certain quantity is extremized.

The most familiar example is that of light, which travels from one point to another by a path taking the shortest amount of time. It turns out that light is not special. All of classical mechanics can be phrased in accordance to this meta-theorem.

1. Hamiltonian mechanics

Let us consider systems whose configurations are described by points in Euclidean space $x \in \mathbb{R}^n$ moving along trajectories $x(t)$. In particular, we might consider a particle in vacuo, a particle attached to a spring or swinging from a pendulum, and so on. Given two points in time t_1 and t_2, which path does our system take, i.e. how does it evolve over time? According to the variational principle above we should find a quantity that is to be extremized by the physical path. We define the **action functional** S as

$$S(x) = \int_{t_1}^{t_2} L(t, x, \dot{x})dt, \tag{1}$$

where L is the **Lagrangian** of our system, which encodes the data of the system at hand. In particular, we will take it to be difference of kinetic and potential energies of the system.

Proposition 1. *An extremal path $x : [t_1, t_2] \to \mathbb{R}^n$ is a solution to the* ***Euler-Lagrange equations***

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j} - \frac{\partial L}{\partial x^j} = 0. \tag{2}$$

Proof. Extremality implies that

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(x + \varepsilon\xi)$$

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for every $\xi : [t_1, t_2] \to \mathbb{R}^n$ with $\xi(t_i) = 0$. Hence

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \left(\sum \frac{\partial L}{\partial x^j} \xi^j + \sum \frac{\partial L}{\partial \dot{x}^j} \dot{\xi}^j \right) dt \\ &= \int_{t_1}^{t_2} \sum \left(\frac{\partial L}{\partial x^j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^j} \right) \xi^j \, dt, \end{aligned}$$

as desired. □

Example 2. As a first check of physical relevance, we show that this variational approach reproduces Newton's second law. Suppose we are given a particle of mass m in $\mathbb{R}^n$ acted upon by a conservative force $F(x) = -\nabla U(x)$, where $U(x)$ is the associated potential energy. Then the Lagrangian for the system can be written

$$L(t, x, \dot{x}) = \frac{1}{2} m |\dot{x}|^2 - U(x), \tag{3}$$

and the Euler-Lagrange equations become:

$$\begin{aligned} 0 &= \frac{d}{dt}(m\dot{x}) - \nabla U \\ &= m\ddot{x} - F, \end{aligned}$$

which is precisely Newton's law, that the force impressed upon an object is proportional to its experienced acceleration. In this picture, Newton's law emerge as the "equations of motion" of our system.

The above formalism for obtaining the equations of motion for a system is known as Lagrangian mechanics. Symplectic geometry emerges when we rewrite the second-order Euler-Lagrange equations as twice as many first-order equations to obtain Hamiltonian mechanics. Introduce the variable

$$p_j = \frac{\partial L}{\partial \dot{x}^j} \tag{4}$$

and define the **Hamiltonian** of a system to be

$$H(t, x, p) = \sum p^j \dot{x}^j - L, \tag{5}$$

where we think of $\dot{x}$, by the implicit function theorem, as a function of the new variable p.[1] The Hamiltonian is actually a very natural quantity: where the Lagrangian is the difference of kinetic and potential energy, the Hamiltonian is actually the sum – the total energy of the system. Moreover, the new variable p can be thought of as momentum. Now it is straightforward to rewrite the Euler-Lagrange equations as **Hamilton's equations**:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \tag{6}$$

Exercise 3. Derive Hamilton's equations from the Euler-Lagrange equations and the definitions above. Recast the example above in Hamiltonian mechanics and verify the physical interpretations just mentioned.

[1] To apply the implicit function theorem we require that the Hessian of L with respect to $\dot{x}^j$ is nonsingular, but we will ignore this technical point.

The motivation for using Hamiltonian mechanics instead of the equivalent Lagrangian mechanics is the following. Take coordinates $z = (x^1, \dots, x^n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$. Then Hamilton's equations above become

$$\frac{dz}{dt} = -\omega_0 \nabla H(t, z), \qquad \omega_0 = \begin{pmatrix} 0 & -\operatorname{id}_{\mathbb{R}^n} \\ \operatorname{id}_{\mathbb{R}^n} & 0 \end{pmatrix}. \tag{7}$$

To put it more geometrically, we have shown that the principle of least action implies that the trajectory or time-evolution of a system is given as the flow along the "Hamiltonian vector field" $-\omega_0 \nabla H(t, z)$. Notice the following two things. First, our geometry is always even dimensional, as we introduce a momentum coordinate for each spatial coordinate. Second, position and momentum are "intertwined" by a skew-symmetric nondegenerate bilinear form ω_0. This is the beginnings of symplectic geometry.

2. Symplectic geometry

Let M be a smooth manifold (without boundary). A **symplectic structure** on a smooth manifold M is the data of a closed, nondegenerate 2-form $\omega \in \Omega^2(M)$. A **symplectic manifold** is a pair (M, ω), where we call ω the "symplectic" form.

Example 4. Let $M = \mathbb{R}^{2n}$ with global coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$. We can equip it with a symplectic form

$$\omega_0 = \sum dx^i \wedge dp_i \tag{8}$$

Exercise 5. Show that every symplectic manifold is even dimensional. Hint: notice that a real skew-symmetric matrix of odd dimension must have a kernel.

A symplectic form satisfies both an algebraic condition (nondegeneracy) as well as a topological condition (ω defines a class in $\mathrm{H}^2(X; \mathbb{R})$). These conditions make symplectic geometry an interesting mixture of "soft" and "rigid". As an example, consider the following surprising theorem:

Theorem 6 (Darboux)**.** *Any two symplectic manifolds are locally symplectomorphic, i.e. locally diffeomorphic with the diffeomorphisms preserving the symplectic forms under pullback.*

Compare this with Riemannian geometry – certainly not every Riemannian manifold of the same dimension is locally isometric! Curvature provides a local invariant of Riemannian manifolds, whereas Darboux's theorem demonstrates that every symplectic manifold (M, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In this sense, symplectic geometry lacks local invariants.

The easiest way to obtain a symplectic manifold is by taking the cotangent bundle of any manifold.

Example 7. Let M be a smooth manifold of dimension n. Then the cotangent bundle $\pi : T^*M$ can be equipped with a symplectic form as follows. Define the **canonical 1-form** $\theta \in \Omega^1(T^*M)$ by

$$\theta_p(v) = \xi(\pi_* v), \tag{9}$$

for $p = (x, \xi)$ and $v \in T_pT^*M$. Then the symplectic form on T^*M is the exterior derivative

$$\omega = -d\theta. \tag{10}$$

Exercise 8. Show that the canonical 1-form is the unique 1-form such that for every $\lambda \in \Omega^1(M)$, $\lambda^*\theta = \lambda$.

Exercise 9. Check that the 2-form defined on T^*M above is indeed nondegenerate. Then verify that the symplectic form defined on $\mathbb{R}^{2n}$ above arises from viewing $\mathbb{R}^{2n}$ as the cotangent bundle of $\mathbb{R}^n$. Hint: nondegeneracy is a local condition, so local coordinates may be helpful.

With these basic definitions and examples in mind, let us return to physics. Let us set up "Hamiltonian mechanics" on an arbitrary symplectic manifold.

Fix a symplectic manifold (M, ω). We should think of M as the **phase space** of our system: the space of all configurations (where "configuration" is meant to refer to both position and momentum, hence $\dim M = n = 2m$). In the example above of a particle in $\mathbb{R}^n$, our phase space was the cotangent space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, where the cotangent directions are thought of as momentum coordinates.

> **Axiom 1:** The configuration space of states of a system is a symplectic manifold.

For any given state in the system, we ought to be able to measure physical quantities of interest. Moreover, these quantities should vary smoothly under small perturbations in state. Thus we define an **observable** of our system simply to be a smooth function $f \in C^\infty(M)$. For our particle, the coordinate functions on $\mathbb{R}^{2n}$ are interesting observables (position and momentum), as is say the energy $H = p^2/2m + U(x)$. According to our physical principles above the **Hamiltonian** $H \in C^\infty(M)$ of our system is distinguished in the sense that it should determine time-evolution. Hence let us denote by (M, ω, H) the data of our Hamiltonian system.

> **Axiom 2:** Observable quantities of a system are smooth functions on the symplectic manifold.

Time-evolution of the system should be given by a symplectomorphism, i.e. a self-diffeomorphism of M pulling back ω to itself. Which one? Well notice that the nondegeneracy of ω uniquely defines a vector field v_H associated to H by

$$dH = \iota_{v_H}\omega, \tag{11}$$

where ι is contraction. We call v_H the **Hamiltonian vector field** for H. We obtain a flow $\phi_H^t : M \to M$ by integrating v_H. One can check that each ϕ_H^t is a symplectomorphism and moreover that v_H is tangent to the level sets of constant energy H. In words, given an initial $t = 0$ configuration $p \in M$, the system is given by $\phi_H^t(p)$ at some later time $t > 0$.

Exercise 10. Check these assertions.

> **Axiom 3:** We distinguish a Hamiltonian $H \in C^\infty(M)$ to determine the time-evolution of the system.

Let's look closer at the differential equation for the flow. By Darboux's theorem we have a chart $V \subset M$ in which the symplectic form is given $\omega = \sum dx^i \wedge dp^i$ with respect to coordinates $(x^1, \ldots, x^n, p^1, \ldots, p^n)$. Then the Hamiltonian vector field for H satisfies

$$dH = \sum \left(\frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial p^i} dp^i \right) = \iota_{v_H} \sum dx^i \wedge dp^i \tag{12}$$

Solving for v_H we find that

$$v_H = \sum \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p^i} \right). \tag{13}$$

The differential equation $\phi_H^t = (x^i(t), p^i(t)) = v_H$ can now be written

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p^i}, \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{14}$$

These are precisely Hamilton's equations! Thus we see that symplectic geometry provides a natural home for Hamiltonian mechanics.

Exercise 11. Suppose (X, g) is a Riemannian manifold. Let $H \in C^\infty(T^*X)$ be given by

$$H(x,p) = \frac{1}{2}|p_x|_g^2, \tag{15}$$

i.e. a purely kinetic term. Show that the Hamiltonian flow on T^*X is dual to the geodesic flow on TX. In other words, the integral curves of v_H project to geodesics of g on X.

Exercise 12. Define the **Poisson bracket** on $C^\infty(M)$ as

$$\{f, g\} = \omega(v_f, v_g). \tag{16}$$

Check that the Poisson bracket gives $C^\infty(M)$ the structure of a Poisson algebra, i.e. a Lie algebra where the Lie bracket is a derivation. A **Poisson manifold** is a smooth manifold together with a Poisson algebra structure on $C^\infty(M)$. Check that time-evolution of a function f is given by $\{H, f\}$.

Show that $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}^*$ is naturally a Poisson manifold via the Lie bracket.

3. Symmetries

Last week Sean showed, in the graduate student seminar, how to (approximately) explain the patterns in the periodic table as an exercise in basic quantum mechanics. The heart of the argument was that the Hamiltonian of the system should be invariant under rotation (as the given system was, after approximation, spherically symmetric). This then implied that the Hilbert space $\mathcal{H}$ of states of the system is a representation of $\mathrm{SO}(3,\mathbb{R})$, and hence must split as irreducibles, from which we deduced the namesake periodic phenomena. This example is just one of many instances where symmetry principles yield powerful methods for attacking problems.

So the last thing I want to discuss is the study of group actions on symplectic manifolds. This is a huge industry, so I will focus on one particular result of interest in physics known as Noether's theorem.

> **Meta-theorem:** Continuous group actions yields conserved quantities.

As specific instances of this result, one might hear a physicist say: "the system is translationally symmetric, hence linear momentum must be conserved," or "the system is time-translationally symmetric, hence energy must be conserved."

In order to make these statements precise, we will need the langague of moment maps.

Definition 13. Suppose a Lie group G acts on (M, ω) through symplectomorphisms. By differentiation, the G-action yields a $\mathfrak{g}$-action, i.e. a vector field $\xi_M \in \Gamma(TM)$ for each $\xi \in \mathfrak{g}$. One can check that the contraction $\iota_{\xi_M}\omega$ is a closed 1-form. This motivates the following definition. We say that the action is **weakly Hamiltonian** if there exists a linear "comoment map" $\kappa : \mathfrak{g} \to C^\infty(M)$ such that

$$\iota_{\xi_M}\omega = d\kappa(\xi) \tag{17}$$

for all $\xi \in \mathfrak{g}$. For weakly Hamiltonian actions we require that the contraction is not only closed but moreover exact.

It is often convenient to repackage this data as follows. A map

$$\mu : M \to \mathfrak{g}^* \tag{18}$$

is a moment map for the G-action if for each $p \in M$,

$$\kappa(\xi)(p) = \langle \mu(p), \xi \rangle, \tag{19}$$

where $\langle \cdot, \cdot \rangle$ is the evaluation pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Using the theory of Lie algebra cohomology one can show the following nontrivial result.

Theorem 14. *Let G act on (M, ω) through symplectomorphisms. If $\mathfrak{g}$ is a semisimple Lie algebra then there exists a (unique) (co)moment map for the action.*

Exercise 15. Let G act on (M, ω) through symplectomorphisms. Show that there is a well-defined linear map

$$\mathrm{H}^1(\mathfrak{g}, \mathbb{R})^* \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \longrightarrow \mathrm{H}^1_{dR}(M, \mathbb{R}) \tag{20}$$

sending $[\xi] \mapsto [\iota_{\xi_M}\omega]$. Show that the action of G is weakly Hamiltonian if and only if this map is identically zero. This presents the cohomological obstruction mentioned above.

We say that a weakly Hamiltonian action is **Hamiltonian** if the comoment map κ is in fact a Lie algebra homomorphism. Show that the G-action defines a cocycle $[\tau] \in \mathrm{H}^2(\mathfrak{g}, \mathbb{R})$ which vanishes if and only if the G-action is Hamiltonian.

Finally, recall that if $\mathfrak{g}$ is a semisimple Lie algebra over a field of characteristic zero and V is any finite-dimensional $\mathfrak{g}$-module then $\mathrm{H}^1(\mathfrak{g}, V) = \mathrm{H}^2(\mathfrak{g}, V) = 0$. This is known as Whitehead's lemma. Using Whitehead's lemma, the theorem follows (though uniqueness takes some work).

This brings us to the following beautiful theorem, due originally to Noether, though first stated in this fashion by Souriau and Smale (I believe).

Theorem 16 (Noether, Souriau, Smale)**.** *Let $\mu : M \to \mathfrak{g}^*$ be a moment map for a weakly Hamiltonian G-action on (M, ω). Then μ is constant along the flow of the Hamiltonian vector field associated to any G-invariant function $H \in C^\infty(M)$.*

Proof. Since $H \in C^\infty(M)^G$, we have

$$H = \psi^*_{\exp(t\xi)} H, \tag{21}$$

for $\xi \in \mathfrak{g}$, where ψ_g is the symplectomorphism given by acting by g. Differentiating this equation at $t = 0$ we obtain,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \psi^*_{\exp(t\xi)} H \\ &= \mathcal{L}_{\xi_M} H = dH(\xi_M) \\ &= \omega(v_H, \xi_M) = -d\kappa(\xi)(v_H) \\ &= -v_H \kappa(\xi). \end{aligned}$$

This shows that $\kappa(\xi)$ is constant along the Hamiltonian flow of H for each $\xi \in \mathfrak{g}$. By definition, $\kappa(\xi)(p) = \langle \mu(p), \xi \rangle$ for $p \in M$, so μ must be constant along the Hamiltonian flow of H as well. □

The statement of the theorem is quite a mouthful, so let's unpack it in the following example.

Example 17. The free one-particle system in $\mathbb{R}^n$ is represented by the symplectic manifold $(T^*\mathbb{R}^n \cong \mathbb{R}^{2n}, \omega)$ and has Hamiltonian $H = |p|^2/2m$. Spatial translation by a group element $v \in \mathbb{R}^n$ is the action of $\mathbb{R}^n$ on $\mathbb{R}^n$ as $x \mapsto x + v$. This action lifts naturally to $T^*\mathbb{R}^n$ as $(x, p) \mapsto (x + v, p)$. The associated Lie algebra action is by $\operatorname{Lie} \mathbb{R}^n \cong \mathbb{R}^n$ as $\xi \mapsto \xi_{T^*\mathbb{R}^3} \sum \xi^i \partial/\partial x^i$. Then

$$\iota_{\xi_{T^*\mathbb{R}^3}} \omega = \left(\sum dx^i \wedge dp^i \right) \xi_{T^*\mathbb{R}^3} = \sum \xi^i dp^i \tag{22}$$

so we obtain a comoment map $\kappa : \operatorname{Lie} \mathbb{R}^n \to C^\infty(T^*\mathbb{R}^n)$ given by

$$\kappa(\xi) = \sum \xi^i p^i. \tag{23}$$

We thus obtain a moment map $\mu : T^*\mathbb{R}^n \to (\operatorname{Lie} \mathbb{R}^n)^*$ determined by

$$\sum \xi^i p^i = \langle \mu(x, p), \xi \rangle. \tag{24}$$

Identifying $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ with the standard Euclidean metric, we conclude that

$$\mu(x, p) = p. \tag{25}$$

By Noether's theorem we find that the linear momentum p is conserved under time-evolution of the system, as the Hamiltonian $H = |p|^2/2m$ is invariant under spatial translation. This is what a physicist means when she says that spatial-translational invariance of a system implies conservation of momentum.

Exercise 18. The free one-particle system in $\mathbb{R}^3$ exhibits rotational invariance. Lift this symmetry to the cotangent bundle and write down the corresponding moment map

$$\mu : T^*\mathbb{R}^3 \longrightarrow \mathfrak{so}(3, \mathbb{R}). \tag{26}$$

Identifying $\mathfrak{so}(3, \mathbb{R})$ with the Lie algebra $(R^3, \times)$, show that

$$\mu(x, p) = x \times p. \tag{27}$$