The Batalin-Vilkovisky Laplacian from homological perturbation theory

Nilay Kumar

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Batalin-Vilkovisky formalism

GAUGE ALGEBRA AND QUANTIZATION

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and

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In respectful memory of Professor Berezin

Quantization of a general gauge theory in the lagrangian approach is accomplished in closed form. The generating equation is found, containing all the relations of the open gauge algebra. A new class of diagrams is revealed, required by BRS-symmetry, but completely definable only from the requirement of unitarity.

Batalin-Vilkovisky formalism

The BV formalism adds to the gauge theory extra fields: *ghosts* and *antifields*. The action is modified

$$S_{\mathrm{BV}} = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$$

to satisfy the *quantum master equation* (QME)

$$\frac{1}{2}(S_{\mathrm{BV}}, S_{\mathrm{BV}}) - i\hbar \Delta S_{\mathrm{BV}} = 0.$$

The QME ensures that the BV functional integrals are well-defined, independent of gauge fixing Lagrangian L:

$$\int_L i_L^*(e^{iS_{\mathrm{BV}}/\hbar}\,\sigma).$$

The geometric context for the BV formalism is odd symplectic geometry.

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The BV Laplacian

In Darboux coordinates $\{x^i, x_i^+\}_{i=1,\dots,n}$ on a finite-dimensional odd symplectic supermanifold (M, ω),

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial x_i^+}, \qquad \Delta^2 = 0.$$

Theorem (Khudaverdian, 2004)

The BV operator $\Delta = \partial^2/\partial x^i \partial x_i^+$ acts covariantly on the half-densities $\Gamma(M, |\Lambda_M|^{1/2})$ of an odd symplectic supermanifold.

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Later, Ševera obtained results linking half-densities to differential forms on M. He gave a spectral sequence construction of Δ.

We present a new, independent proof of Khudaverdian's result:

Theorem (K.)

The BV operator arises locally from homological perturbation theory; that is, transferring the perturbation $d = d_{\mathrm{dR}}$ of the right-hand side of

$$(|\Lambda_M^\hbar|^{1/2}(U), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{dR}_M^\hbar(U), \Omega) \circlearrowleft h$$

yields the BV operator $\hbar\Delta$ on the left. The perturbation setup lifts to Čech complexes, and thus the BV Laplacian globalizes to an operator on the sheaf of half-densities.

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Sign conventions

We work with graded supermanifolds, where coordinates have:

- an internal parity $\mathsf{p}(\phi^i) \in \mathbb{Z}/2\mathbb{Z}$
- an integer grading $\mathsf{gh}(\phi^i) \in \mathbb{Z}$ known as the *ghost number*

The Koszul signs are determined by the total parity

$$|\phi^i| = \mathsf{p}(\phi^i) + \mathsf{gh}(\phi^i).$$

Odd symplectic geometry

A (-1)-shifted odd symplectic form ω on M is a closed two-form providing an isomorphism

$$\omega : TM \to T^*[-1]M$$
$$v \mapsto \omega(v, -)$$

Note: $\omega(v, w) = 0$ unless $\mathrm{gh}(v) + \mathrm{gh}(w) = -1$.

By Darboux's theorem, we can choose coordinates $(x^1, \ldots, x^n, x_1^+, \ldots, x_n^+)$ such that

$$\omega = dx_i^+ \wedge dx^i,$$

where $|x^i| = 0$. We will moreover ask that the body of M is oriented and that the x^i provide an oriented chart for M.

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Ševera's extra differential

The symplectic form ω is odd in the de Rham complex:

$$\mathsf{gh}(\omega) = -1, \mathsf{p}(\omega) = 0, \deg_{\mathrm{dR}}(\omega) = 2 \implies |\omega| = 1$$

and hence squares to zero:

$$\omega^2 = 0.$$

Ševera observed that multiplication by ω,

$$\Omega = \hbar^{-1}\omega \wedge -$$

provides an extra differential on the de Rham complex:

$$\Omega^2 = 0, \qquad [\Omega, d_{\mathrm{dR}}] = 0.$$

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Half-densities

Proposition (Ševera, 2006)

Let (M, ω) be an odd symplectic supermanifold such that the body of M is oriented. Then there is an isomorphism of $\mathcal{O}_M^\hbar$-modules

$$\psi : H^*(\mathrm{dR}_M^\hbar, \Omega) \to |\Lambda_M^\hbar|^{1/2},$$

such that, on a Darboux chart U,

$$\psi_U(f[dx^1 \cdots dx^n]) = f|\mathcal{D}(x, x^+)|^{1/2}.$$

The proof proceeds in two steps:

1. local cohomology computation
2. analysis of the transformation properties

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Cohomology of Ω

Define, on $\mathrm{dR}^{\hbar}_{M}(U)$,

$$\Lambda = \hbar\iota(\partial_{x^i})\iota(\partial_{x_i^+})$$

Notice that

$$\mathsf{gh}(\Lambda) = 1, \mathsf{p}(\Lambda) = 0, \deg_{\mathrm{dR}}(\Lambda) = -2 \implies |\Lambda| = 1.$$

Lemma (Ševera)

The commutator $[\Omega, \Lambda]$ is a semisimple operator on $\mathrm{dR}^{\hbar}_{M}(U)$. For a monomial $\alpha \in \mathrm{dR}^{\hbar}_{M}(U)$,

$$[\Omega, \Lambda]\alpha = (n - \deg_{dx}\alpha + \deg_{dx^+}\alpha)\alpha.$$

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Write $\mathrm{dR}^{\hbar}_M(U)_m$ for the subcomplex of forms with eigenvalue m under $[\Omega, \Lambda]$. Then

$$(\mathrm{dR}^{\hbar}_M(U), \Omega) = \bigoplus_{m=0}^{\infty} (\mathrm{dR}^{\hbar}_M(U)_m, \Omega).$$

Notice that $n - \deg_{dx} + \deg_{dx^+}$ is bounded below by 0, with

$$(\mathrm{dR}^{\hbar}_M(U)_0, \Omega) = (\mathcal{O}^{\hbar}_M(U) \cdot dx^1 \cdots dx^n, 0).$$

Cohomology of Ω

Lemma

The inclusion

$$i : (\mathrm{dR}^{\hbar}_M(U)_0, 0) \hookrightarrow (\mathrm{dR}^{\hbar}_M(U), \Omega).$$

is a quasi-isomorphism.

We build a homotopy $h : \mathrm{dR}^{\hbar}_M(U) \to \mathrm{dR}^{\hbar}_M(U)$,

$$h\alpha = \begin{cases} 0 & \alpha \in \mathrm{dR}^{\hbar}_M(U)_0 \\ m^{-1}\Lambda\alpha & \alpha \in \mathrm{dR}^{\hbar}_M(U)_m, m \neq 0. \end{cases}$$

Then, if p is the projection to $\mathrm{dR}^{\hbar}_M(U)_0$,

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How does dx transform?

Thus the cohomology $H^*(\mathrm{dR}^{\hbar}_M(U), \Omega)$ is generated, on U, by

$$dx^1 \cdots dx^n.$$

In another Darboux coordinate system, $(y^1, \ldots, y^n, y_1^+, \ldots, y_n^+)$, by

$$dy^i = dx^j \frac{\partial y^i}{\partial x^j} + dx_j^+ \frac{\partial y^i}{\partial x_j^+},$$

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$$dy^1 \cdots dy^n = dx^1 \cdots dx^n \det\left(\frac{\partial y^i}{\partial x^j}\right) + \cdots$$

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$$dy^1 \cdots dy^n = dx^1 \cdots dx^n \det\left(\frac{\partial y^i}{\partial x^j}\right) + \cdots$$

involve dx^+ and are Ω-exact. Hence the cohomology classes transform

$$[dy^1 \cdots dy^n] = [dx^1 \cdots dx^n] \left| \det\left(\frac{\partial y^i}{\partial x^j}\right) \right| \qquad (1)$$

according to the inverse determinant of the top-left block of

$$T = \begin{pmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial y^+} \\ \frac{\partial x^+}{\partial y} & \frac{\partial x^+}{\partial y^+} \end{pmatrix}$$

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Half-densities

Lemma (Khudaverdian-Voronov, 2006)

Let A be a symplectic automorphism of an odd symplectic superspace (V, ω). Then

$$\mathrm{Ber}(A) = \det(A_{00})^2,$$

where A_{00} is the even-even block.

The formula

$$[dy^1 \cdots dy^n] = [dx^1 \cdots dx^n] \cdot |\det(T_{00})|^{-1} \tag{2}$$

is precisely the transformation rule for *half-densities*. This completes the proof of Ševera's result.

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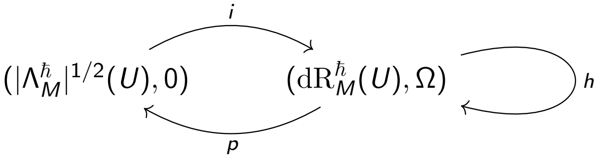
Strong deformation retractions

In the proof of Ševera's result, we have constructed a diagram



such that

$$pi = \mathrm{id}, \qquad \mathrm{id} - ip = [\Omega, h].$$

This data can be systematized in the notion of a strong deformation retraction — this reformulation leads to our construction of Δ.

Strong deformation retractions

Definition

A *strong deformation retraction* (SDR) of complexes is a diagram

$$(C, d_C) \underset{p}{\overset{i}{\rightleftarrows}} (D, d_D) \circlearrowleft h$$

where i and p are maps of complexes and h is a map of degree -1, such that

$$pi = \mathrm{id}_C, \qquad \mathrm{id}_D - ip = [d_D, h],$$

together with the *side conditions*

$$hi = 0, \qquad ph = 0, \qquad h^2 = 0.$$

Homological perturbation lemma

Theorem (Homological perturbation lemma)

Let δ be a small perturbation of d_D, that is, the operator $(\mathrm{id}_D - \delta h)$ is invertible. Then there exists a perturbed strong deformation retraction

$$(C, d'_C = d_C + pAi) \underset{p' = p(1+Ah)}{\overset{i' = (1+hA)i}{\rightleftarrows}} (D, d'_D = d_D + \delta) \circlearrowleft h' = h + hAh$$

where $A = (\mathrm{id}_D - \delta h)^{-1}\delta$.

These formulas were originally discovered by Shih (1962) and Brown (1965) in studying the homology of fiber bundles.

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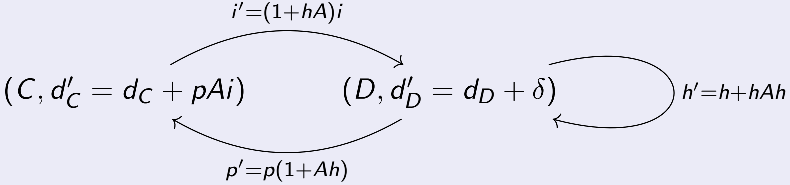

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Local construction of Δ

$$(|\Lambda_M^\hbar|^{1/2}(U), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{dR}_M^\hbar(U), \Omega) \circlearrowleft h$$

Lemma

The de Rham differential $d = d_{\mathrm{dR}}$ is a small perturbation of Ω on the right.

Proof.

The sum

$$(1 - dh)^{-1} = 1 + dh + dhdh + \cdots$$

is finite on any de Rham monomial: $\deg_{\mathrm{dR}} d = 1$ but $\deg_{\mathrm{dR}} h = -2$. □

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Local construction of Δ

Applying the perturbation lemma, we obtain

$$\left(|\Lambda^{\hbar}_M|^{1/2}(U), pAi\right) \underset{p'}{\overset{i'}{\rightleftarrows}} \left(\mathrm{dR}^{\hbar}_M(U), \Omega + d_{\mathrm{dR}}\right) \circlearrowleft h'$$

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The transferred differential on $|\Lambda^{\hbar}_M|^{1/2}(U)$ is the BV Laplacian

$$pAi = \hbar\Delta = \hbar\frac{\partial^2}{\partial x^i \partial x_i^+}.$$

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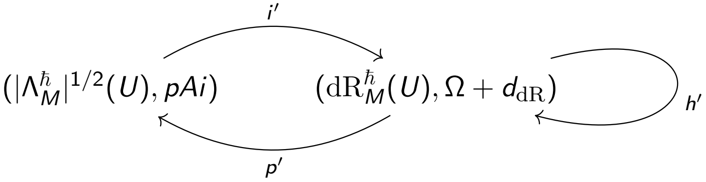

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Proof

Let $\mu = f|\mathcal{D}(x, x^+)|^{1/2} \in |\Lambda_M^\hbar|^{1/2}(U)$. Then

$$\begin{aligned} pAi\mu &= p(1 - dh)^{-1} d(f dx^1 \cdots dx^n) \\ &= p(d + dhd + dhdhd + \cdots) f dx^1 \cdots dx^n \\ &= pdhd(f dx^1 \cdots dx^n) \end{aligned}$$

because p is non-zero only on forms of de Rham degree n.

The remaining term is computed:

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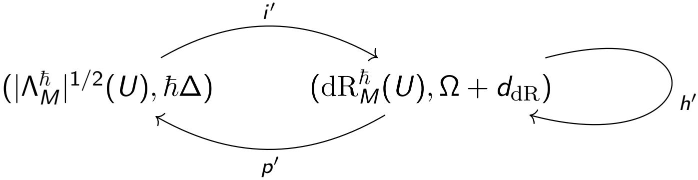
Intertwining Δ and $\Omega + d$



The map i' intertwines the differentials

$$i'(\hbar\Delta\mu) = (\Omega + d_{\mathrm{dR}})i'\mu.$$

Explicitly, if $\mu = f|\mathcal{D}(x, x^+)|^{1/2} \in |\Lambda^\hbar_M|^{1/2}(U)$,

$$i'\mu = \sum_{j=0}^{n} \sum_{k_1<\cdots<k_j} \pm\hbar^j \frac{\partial^j f}{\partial x^+_{k_j} \cdots \partial x^+_{k_1}} dx^1 \cdots \widehat{dx^{k_1}} \cdots \widehat{dx^{k_j}} \cdots dx^n.$$

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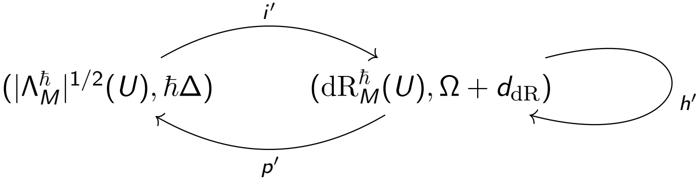

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Globalization

The map $i : |\Lambda^{\hbar}_M|^{1/2}(U) \to \mathrm{dR}^{\hbar}_M(U)$ is coordinate-dependent:

$$dy^1 \cdots dy^n = dx^1 \cdots dx^n \det\left(\frac{\partial y^i}{\partial x^j}\right) + \cdots$$

and so the SDR from before is not an SDR of sheaves.

Our goal is to prove:

Theorem

The local expression $\hbar\,\partial^2/\partial x^i \partial x_i^+$ for the BV Laplacian on half-densities globalizes to a differential on $\Gamma(M, |\Lambda^{\hbar}_M|^{1/2})$.

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Čech complexes

To prove the theorem we upgrade our SDR to a SDR of Čech total complexes:

$$(\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda^\hbar_M|^{1/2}), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}^\hbar_M), \Omega) \circlearrowleft h$$

Here $\mathcal{U}$ is a cover of M by Darboux charts as before, and

$$(\mathrm{Tot}^k \check{C}(\mathcal{U}, |\Lambda^\hbar_M|^{1/2}), 0) = \prod_{p+q=k} \prod_{i_0,\dots,i_p} (|\Lambda^\hbar_M|^{1/2})^q(U_{i_0\cdots i_p}),$$

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The maps i, p, and h are defined on each intersection $U_{i_0 \cdots i_j}$ as in the local case, using the coordinates on U_{i_0}.

$\check{d}$ as a perturbation

The map i needs to be modified in order to intertwine $\check{d}_\Lambda$ with $\check{d} + \Omega$.

Key idea: treat $\check{d}$ as a perturbation of the right-hand side.

Lemma

The Čech differential $\check{d}$ is a small perturbation of Ω; that is, $(\mathrm{id} - \check{d}h)$ is invertible.

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$\check{d}$ as a perturbation

Applying the homological perturbation lemma yields the SDR:

$$\left(\mathrm{Tot}^*\,\check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2}), pAi\right) \underset{p'}{\overset{i'}{\rightleftarrows}} \left(\mathrm{Tot}^*\,\check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar), \Omega + \check{d}\right) \circlearrowleft h'$$

Proposition

The new differential on the left is the Čech differential:

$$pAi = p(\check{d} + \check{d}h\check{d} + \cdots)i = p\check{d}i = \check{d}_\Lambda.$$

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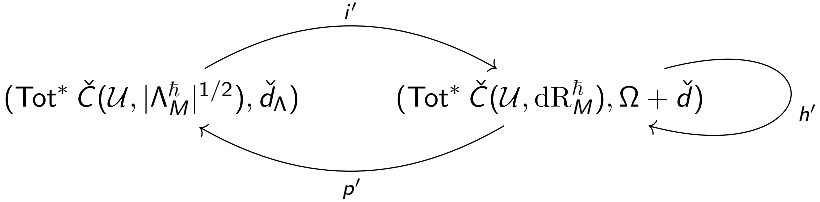
Back to d_{dR}

We now repeat the argument in the local setting.



Perturb the right-hand side by the de Rham differential d_{dR}.

Lemma

The de Rham differential d_{dR} is a small perturbation of $\Omega + \check{d}$; that is, $(\mathrm{id} - d_{\mathrm{dR}} h') = (\mathrm{id} - d_{\mathrm{dR}}(h + hAh))$ is invertible.

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Back to d_{dR}

Applying the perturbation lemma again, we obtain

$$\left(\operatorname{Tot}^* \check{C}(\mathcal{U}, |\Lambda^{\hbar}_M|^{1/2}), \check{d}_\Lambda + p'A'i'\right) \underset{p''}{\overset{i''}{\rightleftarrows}} \left(\operatorname{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}^{\hbar}_M), \Omega + d + \check{d}\right) \circlearrowleft h''$$

The differential $p'A'i'$

The perturbation $p'A'i'$ of $\check{d}_\Lambda$ on the left is:

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The new differential on the left is $\check{d}_\Lambda + pdhdi$.

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Čech 0-cocycles

We now have a new differential $\check{d}_\Lambda + pdhdi$ on the left:

$$\left(\operatorname{Tot}^* \check{C}(\mathcal{U}, |\Lambda^\hbar_M|^{1/2}), \check{d}_\Lambda + pdhdi\right) \underset{p''}{\overset{i''}{\rightleftarrows}} \left(\operatorname{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}^\hbar_M), \Omega + d + \check{d}\right) \circlearrowleft h''$$

The differential $pdhdi$ has Čech degree zero and commutes with $\check{d}$: it sends Čech 0-cocycles to Čech 0-cocycles.

Čech 0-cocycles

Thus *pdhdi* restricts to a well-defined operator on global sections of $|\Lambda^{\hbar}_M|^{1/2}$.

We have already calculated:

$$pdhdi = \hbar \frac{\partial^2}{\partial x^i \partial x_i^+} = \hbar\Delta.$$

Hence we obtain a new proof of Khudaverdian's result:

Theorem (K.)

The differential $\check{d}_\Lambda + pdhdi$ *on* $\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda^{\hbar}_M|^{1/2})$ *restricts to the BV operator* $\hbar\,\partial^2/\partial x^i \partial x_i^+$ *on the global sections* $\Gamma(M, |\Lambda^{\hbar}_M|^{1/2})$ *of the sheaf of half-densities on an odd symplectic supermanifold.*

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More general Lagrangians

Homological perturbation theory gives us explicit formulas for working with half-densities and Δ as differential forms.

Our maps write half-densities locally as

$$\alpha = f(x, x^+) dx^1 \cdots dx^n$$

Notice that α is integrable along the even Lagrangian

$$L = \{x_1^+ = \cdots = x_n^+ = 0\} \subset M.$$

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Differential forms dR^*_M are integrable over submanifolds of odd dimension zero $(k|0)$.

Integral forms Σ^*_M are integrable over sub(super)manifolds of odd *codimension* zero $(k|n)$,

$$\Sigma^*_M = \mathsf{Ber}(M)[0]_{\mathrm{dR}} \otimes_{\mathcal{O}_M} \mathsf{Sym}(TM[1]_{\mathrm{dR}}).$$

Note: Σ^*_M is a dR^*_M-dg-module, and is unbounded below in de Rham degree.

Consider, e.g.

$$f(x, x^+)\mathcal{D}(x, x^+) \otimes (\partial_{x^1})^{a_1} \cdots (\partial_{x^n})^{a_n} (\partial_{x^+_1})^{b_1} \cdots (\partial_{x^+_n})^{b_n}.$$

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Integral forms

Our results from above all hold, *mutatis mutandis*:

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$$|\Lambda^{\hbar}_M|^{1/2} \xrightarrow{\sim} H^*(\Sigma^{\hbar}_M, \Omega)$$
$$f|\mathcal{D}(x, x^+)|^{1/2} \mapsto f\mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}$$

- there is an SDR over which the de Rham differential on $\Sigma^{\hbar}_M$ transfers to the BV operator $\Delta = \hbar\, \partial^2/\partial x^i \partial x_i^+$ on half-densities

Thus: for a purely-odd gauge fixing, the BV operator arises naturally from the de Rham differential on Σ^*_M.

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Future work

Pseudodifferential forms, introduced by Bernstein and Leites, are integrable over arbitrary submanifolds.

Question: Can our methods be extended to the case of pseudodifferential forms?

- allow non-purely-even gauge-fixing with differential forms
- implement Kontsevich-Schwarz dual approach to BV integration
  - L can be thought of as a distributional pseudodifferential form
  - gauge-fixing and the BV integrand on even footing
- functoriality of $(\Gamma(M, |\Lambda_M^\hbar|^{1/2}), \hbar\Delta)$
  - behavior over Lagrangian correspondences $L \hookrightarrow M_1 \times M_2$
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Thank you!