NORTHWESTERN UNIVERSITY

The Batalin-Vilkovisky Laplacian from Homological Perturbation Theory

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

Nilay Kumar

EVANSTON, ILLINOIS

August 2021

ABSTRACT

The Batalin-Vilkovisky Laplacian from Homological Perturbation Theory

Nilay Kumar

The BV Laplacian $\Delta = \partial^2/\partial x^i \partial x_i^+$, first introduced by Batalin and Vilkovisky, is a second-order differential operator that appears in the quantum master equation for quantizing gauge theories. The geometric framework for the BV formalism was later recognized by Schwarz as the setting of odd symplectic geometry and Khudaverdian showed that Δ acts covariantly on half-densities $|\Lambda_M|^{1/2}$ on odd symplectic supermanifolds (M, ω). Building on Ševera's construction of Δ using a spectral sequence for the bicomplex $(\mathrm{dR}_M, \omega + d)$, we provide a new, more explicit construction of Δ using the homological perturbation lemma. The fact that Δ is globally well-defined on half-densities is established using Čech complexes. We show moreover that our methods apply in the setting of integral forms, giving a construction naturally integrable over purely odd Lagrangians.

Acknowledgements

I would like to thank, first and foremost, my advisor Ezra Getzler for his intellectual generosity, support, and encouragement throughout my time at Northwestern. I am also indebted to Kyle Casey, Ryan Contreras, Matei Ionita, Yajit Jain, and Sean Pohorence for valuable discussions and feedback on drafts. I would like to thank Maika Abdallah for sticking with me through thick and thin, through Chicago winters and global pandemics — without her, my work would never have gotten off the ground. Finally, if it were not for the constant support and guidance from my parents, I would not be where I am today. For that I am truly grateful.

This research was supported in part by the National Science Foundation grant "RTG: Analysis on Manifolds" (1502632) at Northwestern University.

Table of Contents

CHAPTER 1

Introduction

The Batalin-Vilkovisky (BV) formalism is a powerful approach to the quantization of physical systems in the presence of gauge symmetry. By adding to the gauge theory certain extra fields known as ghosts and antifields, Batalin and Vilkovisky [1] derive Feynman rules from the functional integral formalism, provided that the modified action S_{BV} satisfies the quantum master equation

$$\frac{1}{2}(S_{\mathrm{BV}}, S_{\mathrm{BV}}) - i\hbar\Delta S_{\mathrm{BV}} = 0. \tag{1.1}$$

Mathematically, the resulting graded supermanifold of fields, ghosts, and antifields is a shifted cotangent bundle $M = T^*[-1]X$ equipped with an odd symplectic form ω of degree -1. The antibracket $(-,-)$ is the odd Poisson bracket associated to ω, but the origin of the BV operator Δ is more subtle.

In Darboux coordinates $\{x^i, x_i^+\}_{i=1,\dots,n}$ on the shifted cotangent bundle, the BV operator resembles a Laplacian,

$$\Delta = \frac{\partial^2}{\partial x^i \partial x_i^+}, \tag{1.2}$$

where we sum implicitly over the index i. Unlike the ordinary Laplacian, Δ does not act on functions of M. Instead, as discovered by Khudaverdian, Δ is intrinsically defined if we view it as acting on half-densities.

Theorem 1 (Khudaverdian [11]). The BV operator $\Delta = \partial^2/\partial x^i \partial x_i^+$ acts covariantly on the half-densities $|\Lambda_M|^{1/2}$ of an odd symplectic supermanifold.

This perspective elucidates the geometry of the BV formalism: the integrand of the BV functional integral is a half-density, gauge-fixing is performed by choosing a Lagrangian to integrate this half-density over, and the independence of choice of gauge is guaranteed by the quantum master equation.

The antibracket arises naturally as a global operation from the odd symplectic structure on the space of fields; the BV operator Δ, in contrast, is defined locally and then checked to be independent of the choice of coordinates on half-densities. Khudaverdian verifies invariance by classifying the canonical transformations of Darboux coordinates on odd symplectic manifolds and computing the corresponding transformation rules for Δ. It is compelling to ask for a less ad hoc derivation of the BV Laplacian — one that unearths Δ directly from the odd symplectic structure of the shifted cotangent bundle.

To this end, Ševera in [17] provides a homological description of Δ that originates from the presence of an extra differential on the complex of de Rham forms dR_M. As ω is an odd symplectic form, the operator $\omega = \omega \wedge -$ squares to zero and (anti)commutes with the de Rham differential:

$$\omega^2 = 0, \qquad [\omega, d] = 0. \tag{1.3}$$

From the spectral sequence associated to this bicomplex, Ševera finds the BV operator as a differential on the third page, acting on the cohomology of ω. The cohomology $H^*(\mathrm{dR}_M, \omega)$ he in turn identifies with the space of half-densities $|\Lambda_M|^{1/2}$ on M, thus recovering the result of Khudaverdian.

In this thesis we take a more direct approach to the BV Laplacian, replacing spectral sequences by homological perturbation theory. This allows us to write explicit maps and formulas relating half-densities to differential forms. The key idea is to set up, on any Darboux coordinate chart U, a strong deformation retraction of complexes (see Definition 14)

$$(|\Lambda_M|^{1/2}(U), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{dR}_M(U), \omega) \circlearrowleft h \qquad \mathrm{id} - ip = [\omega, h] \tag{1.4}$$

and to treat the de Rham differential $d = d_{\mathrm{dR}}$ as a perturbation of the differential ω on the right. The homological perturbation lemma induces a perturbation on the left — the transfer of the perturbation d — as a certain formal infinite sum. We show that this sum is finite, which leads to our first result:

Theorem 2. For (M,ω) an odd symplectic supermanifold with $\mathrm{gh}(\omega) = -1$, there exists a strong deformation retraction

$$(|\Lambda_M|^{1/2}(U), \Delta) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathrm{dR}_M(U), \omega + d) \circlearrowleft h'$$

where the differential Δ on the left is the BV operator.

In particular, there exists a locally-defined map i' that identifies half-densities as differential forms in such a way that $i'(\Delta\mu) = (\omega + d)i(\mu)$.

The formulas used to construct this strong deformation retraction are coordinate-dependent, and thus do not immediately yield a global construction of Δ. We proceed by choosing a Darboux atlas $\mathcal{U}$ of M and promoting the the strong deformation retraction of Equation 1.4 to a strong deformation retraction of Čech total complexes. The map i does not commute with the Čech differential $\check{d}$, so we introduce $\check{d}$ as a perturbation of the differential on the right and apply homological perturbation theory once more:

$$(\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda_M|^{1/2}), \check{d}) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathrm{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}_M), \pm\,\omega + \check{d}) \circlearrowleft h'$$

We are left with a modified map i' that does indeed intertwine the desired differentials. We can now imitate the local construction and perturb the differential on the right by $d = d_{\mathrm{dR}}$, transferring it to the left to obtain the operator Δ. As a perturbation of the Čech differential on the left, Δ commutes with $\check{d}$. This is precisely the statement that Δ globalizes, and we obtain a new, independent construction of Khudaverdian's BV operator.

Theorem 3. The operator $\Delta = \partial^2/\partial x^i \partial x_i^+$ on half-densities constructed by homological perturbation theory over a Darboux coordinate chart extends to a well-defined operator on the sheaf of half-densities.

The diagrams above identify half-densities with differential forms written

$$f(x, x^+)\, dx^1 \cdots dx^n,$$

where the x^i are the even Darboux coordinates. From the perspective of gauge-fixing in physics, these are precisely the forms that can be integrated over even Lagrangians. There is no fundamental reason to restrict to this class of Lagrangians, but the above formalism is not appropriate in the general case, as differential forms are not the natural objects for integration over general supermanifolds. For a purely odd Lagrangian, for instance, it is necessary to consider *integral forms* [4], and for a general Lagrangian, *pseudodifferential forms* [3]. We show that the BV operator can be constructed as before if we replace the right-hand side of the strong deformation retraction with the complex of integral forms Σ_M.

Theorem 4. With (M, ω) as before, there exists a strong deformation retraction

$$(|\Lambda_M|^{1/2}(U), \Delta) \underset{p'}{\overset{i'}{\rightleftarrows}} (\Sigma_M(U), \omega + d_\Sigma) \circlearrowleft h'$$

where the differential Δ on the left is the BV operator.

The global perturbation argument in the case of integral forms proceeds identically as described above for differential forms.

The corresponding homological relationship between pseudodifferential forms and half-densities is more subtle, and we do not pursue it here. We note, however, that obtaining formulas in the general setting of pseudodifferential forms is of interest for a number of reasons beyond simply the presence of Lagrangians of general type. Following an unpublished idea of Kontsevich and Schwarz, we might approach the BV formalism from the viewpoint of duality. That is, we might view the Lagrangian itself as a distributional form with support along the

Lagrangian, and BV integration as a special case of the pairing between distributional and smooth pseudodifferential forms. From this perspective, the choices of BV action and gauge-fixing Lagrangian are placed on similar footing.

From a more categorical perspective, we note that the results of this paper are functorial with respect to symplectomorphisms. In many situations, however, we are more interested in maps between fundamentally distinct spaces of fields, such as those constructed by symplectic reduction. Weinstein's notion of a Lagrangian correspondence allows us to study such maps by considering spans $M_1 \leftarrow L \rightarrow M_2$ with $L \subset M_1 \times M_2$ Lagrangian. The functoriality of half-densities over such correspondences is not immediately obvious, whereas pseudodifferential forms enjoy natural pullback and pushforward (fiber integration) operations.

Unfortunately, the methods we use here do not obviously generalize to pseudodifferential forms and Lagrangians of arbitrary type. As we note in Chapter 5, the local results of this thesis hold for a certain subclass of pseudodifferential forms, but we leave a more detailed study for future work.

Conventions

We note several typographical and sign conventions that will be adhered to throughout, unless otherwise indicated.

- We follow the Einstein summation convention: all repeated upper-lower index pairs are implicitly summed over.
- Our underlying geometric objects of study are smooth, finite-dimensional, $\mathbb{Z}$-graded supermanifolds $(M, \mathcal{O}_M)$. In particular, any coordinate x is assigned a $\mathbb{Z}/2\mathbb{Z}$-graded internal parity $\mathrm{p}(x)$ and a $\mathbb{Z}$-graded ghost number $\mathrm{gh}(x)$. These are independent gradings: the $\mathbb{Z}$-grading *need not* reduce to the $\mathbb{Z}/2\mathbb{Z}$-grading modulo 2. We write $|x|$ for the total parity
$$|x| = \mathrm{p}(x) + \mathrm{gh}(x) \pmod 2$$
which determines the choice of any Koszul signs. The body of a supermanifold M is the ordinary smooth manifold $(|M|, \mathcal{O}_M/\mathcal{J}_M)$, where $\mathcal{J}_M$ is the ideal of nilpotent elements of $\mathcal{O}_M$.
- We work with cohomologically-graded complexes of real superspaces, with the $\mathbb{Z}$-grading given by ghost number and the $\mathbb{Z}/2\mathbb{Z}$-grading given by parity. When discussing de Rham (resp. integral) forms, there is an auxiliary $\mathbb{Z}$-grading by de Rham degree, with respect to which we implicitly take a direct sum total complex; that is, the $\mathbb{Z}$-grading is the sum of the ghost number and the de Rham degree. The total parity of a homogeneous de Rham (resp. integral) form is thus written
$$|\alpha| = \mathrm{p}(\alpha) + \mathrm{gh}(\alpha) + \deg_{\mathrm{dR}}(\alpha) \pmod 2.$$
We denote by $C[j]^*$ the shift of the complex C^* by j in the ghost number grading, with $C[j]^k = C^{j+k}$, and a shift by j in the de Rham grading by $C\{j\}^k = C^{j+k}$.

CHAPTER 2

Odd symplectic structures

The natural geometric setting for the BV formalism is that of degree -1 odd symplectic graded supergeometry. In this chapter we review some of the fundamental results in this context, following Khudaverdian and Voronov [12], Getzler and Pohorence [9] and Ševera [17]. The basic notions of the theory of supermanifolds are well-established (see for instance [13] and [14]), but we will recall the definition of the Berezinian (or the superdeterminant), as it plays an important role in the odd symplectic world.

We begin with some linear superalgebra, referring to Section 7 of Leites [13] and Chapter 3 of Manin [14] for details. Let V be a finite-dimensional real superspace, $V = V_0 \oplus V_1$. An endomorphism $T \in \mathrm{End}(V)$ can be written as a matrix

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix},$$

where

$$T_{00} : V_0 \to V_0 \qquad T_{01} : V_1 \to V_0,$$
$$T_{10} : V_0 \to V_1 \qquad T_{11} : V_1 \to V_1.$$

Then, for $T \in \mathrm{End}(V)$, the *Berezinian* of T is

$$\mathrm{Ber}\, T = \mathrm{Ber} \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} = \det(T_{00} - T_{01} T_{11}^{-1} T_{10}) \det T_{11}^{-1} \in \mathbb{R}^{\times}.$$

If V is purely even, the Berezinian reduces to the determinant, whereas if V is purely odd, the Berezinian reduces to the inverse of the determinant. Like the determinant, the Berezinian is a homomorphism and hence $\mathrm{Ber}\, TT' = \mathrm{Ber}\, T\, \mathrm{Ber}\, T'$. Note, however, that while the determinant is a polynomial in the entries of the matrix, the Berezinian is a rational function in general.

We write $\mathrm{GL}(V) \subset \mathrm{End}(V)$ for the general linear supergroup of endomorphisms with well-defined, non-zero Berezinian. If $\dim V_0 = p$ and $\dim V_1 = q$, we will often write more explicitly $\mathrm{GL}(V) = \mathrm{GL}(p|q)$.

Suppose now that we have on V a non-degenerate antisymmetric bilinear pairing of odd parity, that is, an odd symplectic structure ω,

$$\omega : V \times V \to \mathbb{R}.$$

Notice that ω is non-zero only when pairing even vectors with odd vectors, and hence $p = q = n$. Recall that a Lagrangian subspace $L \subset V$ is an isotropic subspace, $\omega|_L = 0$, of maximal dimension n. A *polarization* of V is a decomposition

$$V = L \oplus L'$$

into two complementary Lagrangian subspaces. The form ω provides an isomorphism $L' \cong \Pi L^*$. Abbreviating

$$L^\circ = \Pi L^*,$$

we may choose a basis for L and a dual, parity-inverted basis for L°, and decompose any $T \in \mathrm{GL}(n|n)$ as

$$T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where $P : L \to L$, $Q : L^\circ \to L$, $R : L \to L^\circ$, and $S : L^\circ \to L^\circ$.

Proposition 5 (Khudaverdian and Voronov [12], Theorem 4)**.** Let $V = L \oplus L^\circ$ be a polarization with respect to the odd symplectic form ω on V. If $T \in \mathrm{GL}(V)$ preserves the odd symplectic form — that is, $T \in \Pi\,\mathrm{Sp}(V, \omega)$ — then

$$\mathrm{Ber}\, T = (\mathrm{Ber}\, P)^2. \tag{2.1}$$

Proof. Choose a basis $\{e_1, \dots, e_n\}$ of L. This basis can be completed to a basis of V by finding $f_i \in V$ such that $\omega(e_i, f_j) = \delta_{ij}$. If we denote the dual basis of V^* by $\{E_i, F_i\}$, we find that ω maps e_i to F_i and f_i to $-E_i$. Hence under the decomposition $(V)^\circ \cong (L \oplus L^\circ)^\circ \cong L^\circ \oplus L$

we can write

$$\omega = \begin{pmatrix} \mathrm{id}_L & 0 \\ 0 & -\mathrm{id}_{L^\circ} \end{pmatrix}.$$

The requirement that T preserve the odd symplectic form,

$$T^\circ \omega T = \omega,$$

is therefore

$$\begin{pmatrix} S^\circ & Q^\circ \\ R^\circ & P^\circ \end{pmatrix} \begin{pmatrix} \mathrm{id}_L & 0 \\ 0 & -\mathrm{id}_{L^\circ} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \mathrm{id}_L & 0 \\ 0 & -\mathrm{id}_{L^\circ} \end{pmatrix},$$

which we may rewrite as

$$\begin{aligned} S^\circ P &= Q^\circ R + \mathrm{id}_L \\ Q^\circ S &= S^\circ Q \\ P^\circ R &= R^\circ P \\ P^\circ S &= R^\circ Q + \mathrm{id}_{L^\circ}. \end{aligned}$$

Factoring

$$T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \mathrm{id}_L & 0 \\ RP^{-1} & \mathrm{id}_{L^\circ} \end{pmatrix} \begin{pmatrix} P & Q \\ 0 & S - RP^{-1}Q \end{pmatrix}$$

we obtain

$$\mathrm{Ber}\, T = \mathrm{Ber}\, P\, \mathrm{Ber}(S - RP^{-1}Q).$$

The third and fourth equations from above show that

$$P^\circ S - R^\circ Q = P^\circ S - P^\circ R P^{-1} Q = \mathrm{id}_{L^\circ}$$

from which we find

$$\mathrm{Ber}(S - RP^{-1}Q) = (\mathrm{Ber}\, P^\circ)^{-1} = \mathrm{Ber}\, P.$$

The result follows. □

We may think of this result as the odd symplectic equivalent of the fact that the determinant of an ordinary symplectic matrix is 1. Indeed, the proof in the even case proceeds *mutatis mutandis* and the calculation reduces to a product $\det P \det(S - RP^{-1}Q) = \det P \det P^{-1} = 1$. In the odd setting the product is instead a quotient, as is characteristic of the Berezinian, giving a perfect square.

We define, for $T \in \Pi\,\mathrm{Sp}(V, \omega)$,

$$|\,\mathrm{Ber}^{1/2}(T)| = |\,\mathrm{Ber}(P)|,$$

where P is the upper-left block of T with respect to any polarization of V. This definition is independent of the choice of polarization and yields a character of the odd symplectic supergroup $\Pi\,\mathrm{Sp}(V, \omega)$ of automorphisms preserving ω:

$$|\,\mathrm{Ber}^{1/2}(TT')| = |\,\mathrm{Ber}^{1/2}(T)||\,\mathrm{Ber}^{1/2}(T')|.$$

Remark 6. Choosing the Lagrangian L in the polarization of Proposition 5 to be purely even and denote by P the resulting upper left block of T, we find that

$$\mathrm{Ber}\,T = \mathrm{Ber}\,P^2 = \det P^2.$$

This shows, in particular, that the Berezinian of an odd symplectic matrix T is a polynomial in the entries of T.

Leaving the world of linear superalgebra, we now turn to odd symplectic geometry. Let us first review what we mean by a smooth graded supermanifold. There are a number of different definitions in the presence of both a $\mathbb{Z}$ and a $\mathbb{Z}/2\mathbb{Z}$-grading — our conventions line up with those used, for example, by Voronov [19]. The sheaf of commutative algebras O_M on a smooth supermanifold M is locally isomorphic to the tensor product of smooth functions on a vector space W with the exterior algebra of a vector space V^*. We say that M is a graded supermanifold when W and V are $\mathbb{Z}$-graded vector spaces and there exists an atlas of M such that the transition functions respect this grading (or ghost number, as it is known in the physics literature). Moreover, as mentioned in the conventions above, the $\mathbb{Z}$-grading is independent of the $\mathbb{Z}/2\mathbb{Z}$-grading in general, and the relevant Koszul signs that appear are based wholly on the $\mathbb{Z}/2\mathbb{Z}$-grading. That is, signs appear according to what we call in this thesis *total parity*. We will often refer to the *internal parity* of an object: this is the difference between the $\mathbb{Z}/2\mathbb{Z}$-degree and the mod 2 reduction of the $\mathbb{Z}$-degree. From the perspective of physics, this allows for the presence of physical fermions — that is, coordinates of $\mathbb{Z}$-degree 0 but odd internal parity, and therefore odd total parity — as well as non-polynomial functions of $\mathbb{Z}$-degree different than 0, such as exponentiated actions.

The de Rham complex dR_M is a sheaf of bigraded commutative O_M-superalgebras on M generated by Kähler differentials placed in de Rham degree 1 (recall that $\{-\}$ represents a shift in de Rham degree),

$$(\mathrm{dR}_M, d) = (\mathrm{Sym}_{O_M}(T^*M\{-1\}), d = d_{\mathrm{dR}}),$$

equipped with the usual differential $df = dx^i \partial_{x^i} f$. Abusing notation slightly, we will continue to denote by dR_M the direct sum total complex of this bigraded complex. That is, if $f \in O_M$, then the total degree of df is $\mathrm{gh}(df) + \deg_{\mathrm{dR}}(df) = \mathrm{gh}(f) + 1$ while the internal parity $\mathrm{p}(df) = \mathrm{p}(f)$ remains unchanged. Hence df has total parity opposite to that of f: $|df| = |f| + 1$.

We warn the unfamiliar reader that the de Rham complex on supermanifolds is unbounded above in general. If θ^i is a coordinate with odd total parity, for instance, then $d\theta^i$ has even total parity, and thus $(d\theta^i)^k$ is non-zero for arbitrarily high k. Consequently, there is no notion of top-degree volume form in the de Rham complex of a supermanifold.

There is a link — discovered by Ševera in [17] — between differential forms and half-densities on odd symplectic supermanifolds. The Berezinian bundle $\mathrm{Ber}(M) = \mathrm{Ber}(T^*M)$ of M is the bundle associated to the character $\mathrm{Ber}(T)^{-1}$ of $\mathrm{GL}(p|q)$. Similarly, the bundle $|\Lambda_M|$ of *densities* on M is associated to the character $|\mathrm{Ber}(T)|^{-1}$ and the bundle $|\Lambda_M|^{1/2}$ of *half-densities* on M is associated to the character $|\mathrm{Ber}^{1/2}(T)|^{-1}$. To explain the connection with differential forms we first detail the basic definitions of odd symplectic geometry.

For the rest of this thesis, M will be a graded supermanifold of dimension $n|n$ equipped with an odd symplectic form ω of degree -1. We will moreover assume that the body $|M|$ of M is connected and oriented.[1] By an odd symplectic form ω of degree -1, we mean a closed form $\omega \in \Gamma(M, \mathrm{dR}_M)$ of de Rham degree 2, ghost number -1 (hence of total $\mathbb{Z}$-degree $+1$), and even internal parity (hence odd total parity), satisfying a nondegeneracy condition. The form ω pairs

[1]The condition that $|M|$ is oriented may be removed if we work throughout with *half-forms* rather than half-densities.

tangent vectors v and v' nontrivially only if $\mathrm{gh}(v) + \mathrm{gh}(v') = -1$ and as such, provides a map

$$\omega : TM \to T^*[-1]M$$
$$v \mapsto \omega(v, -)$$

where $T^*[-1]M$ is the (-1)-shifted cotangent bundle whose fibers are modeled on $(V[-1])^* = V^*[1]$ if M is modeled on a superspace V. The nondegeneracy condition is, as usual, that this map be an isomorphism of bundles.

The proof of the Darboux theorem in the context of ordinary symplectic geometry transfers *mutatis mutandis* to the the odd symplectic setting. It will be useful to stipulate two conditions on our choice of Darboux charts.

Definition 7. Let (M, ω) be an odd symplectic supermanifold such that the body $|M|$ is oriented. An *oriented Darboux chart* consists of a Darboux chart $U \subset M$ with coordinates $\{x^i, x_i^+\}_{i=1,\dots,n}$ such that the x^i have even total parity and provide a positively oriented chart for the body $|M|$.

The condition that the x^i have even total parity can be arranged by suitable choice of symplectic basis in the proof of the Darboux theorem. Then, by negating x^1 and x_1^+ if necessary, any such Darboux coordinates can be modified to produce an oriented Darboux chart. We remark that this choice of parity for the x^i is generally not physical: a physical fermion (a coordinate of ghost number 0 and odd internal parity), for instance, is not usually viewed as an antifield. We nevertheless choose this polarization for calculational ease; the choice of a more physically conventional polarization introduces signs but does not change the results of this thesis.

The symplectic form is written locally as

$$\omega = dx_i^+ \wedge dx^i.$$

The form is exact, as $\omega = d\lambda$, where $\lambda = x_i^+ dx^i$. As ω has odd total parity in the de Rham complex, $\omega^2 = 0$, and the operator

$$\Omega = \omega \wedge -$$

provides an extra differential on the de Rham complex. It increases de Rham degree by 2, decreases the ghost number by 1, and leaves the internal parity invariant, but is not compatible with the wedge product. The symplectic form ω is closed, so the de Rham differential commutes with Ω,

$$[d, \Omega] = 0,$$

and we obtain the total differential $\Omega + d_{\mathrm{dR}}$.

Ševera's insight in [17] was to identify the cohomology of Ω with the space of half-densities on M.

Proposition 8 (Ševera [17], Section 2)**.** Let (M, ω) be an odd symplectic supermanifold with $\mathrm{gh}(\omega) = -1$ such that the body $|M|$ of M is oriented. Then there is a natural isomorphism of sheaves

$$\psi : H(\mathrm{dR}_M, \Omega) \to |\Lambda_M|^{1/2}, \tag{2.2}$$

between the cohomology of the differential Ω and the half-densities on M.

Remark 9. There is no clear consensus in the literature on the conventions for the parities and gradings of $|\Lambda_M|$ and $|\Lambda_M|^{1/2}$. In this thesis, we will adopt a convention suggested by the proof Proposition 8 (and the results that follow). Write

$$r = n + \sum_{i=1}^{n} \mathrm{gh}(x^i).$$

We place the Berezinian and the bundle of densities of M in ghost number $2r$ with even internal parity, and the bundle of half-densities in ghost number r with internal parity $\sum_{i=1}^{n} \mathrm{p}(x^i)$ (and hence total parity n).

With respect to the conventions of Remark 9, the map ψ has ghost number 0 and even internal parity, as the representative $dx^1 \cdots dx^n$ has de Rham degree n, ghost number $\sum_{i=1}^{n} \mathrm{gh}(x^i)$, and parity $\sum_{i=1}^{n} \mathrm{p}(x^i)$.

The proof of Proposition 8 proceeds by computing the cohomology explicitly in local coordinates, and then showing that the resulting generator transforms as a half-density. We will prove this result in detail in the remainder of this chapter, as the constructions involved will be useful later for building the BV operator.

The significance of half-densities on odd symplectic supermanifolds comes from the fact that they can be naturally integrated over Lagrangian submanifolds.

Proposition 10. Let $i_L : L \hookrightarrow M$ be a Lagrangian submanifold of (M, ω). Then the bundle of half-densities on M restricts to the bundle of densities on L,

$$i_L^* |\Lambda_M|^{1/2} \cong |\Lambda_L|.$$

This is a direct consequence of Equation 2.1, and is, in part, why half-densities hold a particular significance in the BV formalism: the choice of a Lagrangian is the choice of a gauge-fixing, and integration along Lagrangians is used to define correlation functions. This geometric interpretation of the work of Batalin and Vilkovisky [1] is due originally to Schwarz [16] and was rephrased in terms of half-densities by Khudaverdian [11].

Before we investigate the cohomology of Ω, we make a small change of notation. In the physics literature, there is typically a formal parameter $\hbar$ employed in the quantum BV formalism. This parameter measures, loosely, the distance between the quantum theory and its classical counterpart. Thus we will work with functions on M that are Laurent series in $\hbar$,

$$\mathcal{O}_M^\hbar = \mathcal{O}_M((\hbar)),$$

with $\hbar$ having ghost number zero and even internal parity. We write similarly $\mathrm{dR}_M^\hbar$ and $|\Lambda_M^\hbar|^{1/2}$ for the $\mathcal{O}_M^\hbar$-modules of de Rham forms and half-densities, respectively. We scale the differential ω on the de Rham complex by $\hbar^{-1}$,

$$\Omega = \hbar^{-1}\omega \wedge -.$$

This particular scaling is chosen, as we will see, in order for the BV Laplacian to be proportional to $\hbar$, as is standard in the BV formalism.

Remark 11. We assume throughout this thesis, that ω has ghost number -1: it thus follows that $\Omega = \hbar^{-1}\omega$ is a (even internal parity) differential of total degree 1 on the de Rham complex. In other contexts such as the BFV formalism or AKSZ sigma models it is sometimes useful to take different $\mathrm{gh}(\omega) \in \mathbb{Z}$. We remark that in these settings, if $\mathrm{gh}(\omega) = 2k - 1$ is odd (still with even internal parity), we may take the parameter $\hbar$ to have ghost number $2k$ such that $\hbar^{-1}\omega$ has ghost number -1 and is again a differential of total degree 1. In this case, as $\hbar$ has non-zero degree we must be careful to work with Laurent polynomials in $\hbar$ instead of Laurent series.

Shifted symplectic forms also feature in the work of Pantev, et. al (PTVV) [15]. The geometric setting there is that of derived stacks, in which the internal degree present on the structure sheaf allows for the notion of symplectic forms of shifted degree. PTVV demonstrate that a large class of derived stacks of interest (such as certain classifying spaces and Lagrangian intersections) are equipped with natural symplectic structures of various degree shifts. They show moreover how to construct symplectic structures on mapping stacks from those on the target (with the new shift computed from the dimension of the domain), which provides a procedure for building new shifted symplectic structures from old. Though the geometric setting of PTVV is quite different than ours, it is useful to keep in mind the ideas in the derived algebraic setting.

We now turn to the computation of the cohomology $H(\mathrm{dR}^{\hbar}_{M}, \Omega)$. For the rest of this chapter we work in an oriented Darboux chart U with coordinates (x^i, x^+_i). Following Ševera, we define the operator

$$\Lambda = \hbar\, \iota(\partial_{x^i})\iota(\partial_{x^+_i})$$

on $\mathrm{dR}^{\hbar}_{M}(U)$. Notice that Λ has odd total parity, as it decreases de Rham degree by 2, increases ghost number by 1, and has even internal parity. The importance of Λ stems from the following diagonalization result on U.

Lemma 12. The commutator $[\Omega, \Lambda]$ is a semisimple operator. Indeed, given a monomial $\alpha \in \mathrm{dR}^{\hbar}_{M}(U)$, the commutator acts as

$$[\Omega, \Lambda]\alpha = \left(n - \deg_{dx} \alpha + \deg_{dx+} \alpha\right) \alpha,$$

giving us an explicit basis of eigenvectors. The eigenvalues are non-negative, and the 0-eigenspace is generated by $dx^1 \cdots dx^n$.

Proof. Note first that $[\Omega, \Lambda] = \Omega\Lambda + \Lambda\Omega$. We compute the second term using the Leibniz rule and the appropriate Koszul signs:

$$\begin{aligned}\Lambda\Omega\alpha &= \left(\iota(\partial_{x^\ell})\iota(\partial_{x^+_\ell})(dx^k dx^+_k)\right)\alpha + \left(\iota(\partial_{x^\ell})(dx^k dx^+_k)\right)\iota(\partial_{x^+_\ell})\alpha \\ &\quad - \left(\iota(\partial_{x^+_\ell})(dx^k dx^+_k)\right)\iota(\partial_{x^\ell})\alpha - dx^k dx^+_k \iota(\partial_{x^\ell})\iota(\partial_{x^+_\ell})\alpha \\ &= \left(\delta^k_\ell \delta^\ell_k + dx^+_\ell \iota(\partial_{x^+_\ell}) - dx^\ell \iota(\partial_{x^\ell}) - \Omega\Lambda\right)\alpha \\ &= \left(n + \deg_{dx^+}\alpha - \deg_{dx}\alpha - \Omega\Lambda\right)\alpha.\end{aligned}$$

To see that the lowest eigenvalue is 0, we note that $\deg_{dx}$ is maximized at n when the differentials of all the even coordinates are present. □

Write $\mathrm{dR}^\hbar_M(U)_m$ for the space of de Rham forms with $[\Omega, \Lambda]$-eigenvalue m. The operator Ω commutes with $[\Omega, \Lambda]$,

$$[\Omega, [\Omega, \Lambda]] = \frac{1}{2}[[\Omega, \Omega], \Lambda] = [\Omega^2, \Lambda] = 0,$$

so Lemma 12 yields a splitting of complexes

$$(\mathrm{dR}^\hbar_M(U), \Omega) = \bigoplus_{m=0}^{\infty} (\mathrm{dR}^\hbar_M(U)_m, \Omega),$$

where the $m = 0$ eigenspace is one-dimensional, written

$$(\mathrm{dR}^\hbar_M(U)_0, \Omega) = (\mathcal{O}^\hbar_M(U) \cdot dx^1 \cdots dx^n, 0). \tag{2.3}$$

Notice that Ω restricted to the 0-eigenspace is zero:

$$\Omega\, dx^1 \cdots dx^n = \hbar^{-1}\omega \wedge dx^1 \cdots dx^n = 0. \tag{2.4}$$

We now show that the inclusion

$$i : (\mathrm{dR}^\hbar_M(U)_0, 0) \hookrightarrow (\mathrm{dR}^\hbar_M(U), \Omega)$$

is a quasi-isomorphism and hence that $dx^1 \cdots dx^n$ generates the cohomology of Ω. Write

$$p : (\mathrm{dR}^\hbar_M(U), \Omega) \to (\mathrm{dR}^\hbar_M(U)_0, 0)$$

for the projection onto the $m = 0$ eigenspace of $[\Omega, \Lambda]$. Note that i and p are maps of complexes by Equation 2.4. We define a homotopy

$$h : \mathrm{dR}^{\hbar}_M(U) \to \mathrm{dR}^{\hbar}_M(U)$$

of total degree -1 and even internal parity as follows: for $\alpha \in \mathrm{dR}^{\hbar}_M(U)_m$,

$$h(\alpha) = \begin{cases} m^{-1}\Lambda\,\alpha & m \neq 0 \\ 0 & m = 0. \end{cases} \tag{2.5}$$

As the next following lemma shows, the complexes $\mathrm{dR}^{\hbar}_M(U)_m$ for $m \neq 0$ are all contractible.

Lemma 13. The cohomology of Ω is generated by $dx^1 \cdots dx^n$:

$$H(\mathrm{dR}^{\hbar}_M(U), \Omega) \cong (\mathrm{dR}^{\hbar}_M(U)_0, 0).$$

Proof. We show that i is a quasi-isomorphism. It suffices to check that h is indeed a homotopy, that is,

$$\mathrm{id} - i \circ p = [\Omega, h].$$

Suppose $\alpha \in \mathrm{dR}^{\hbar}_M(U)_m$ for $m \neq 0$. Then $i(p(\alpha)) = 0$ while

$$[\Omega, h]\alpha = \Omega \cdot m^{-1}\Lambda\alpha + m^{-1}\Lambda\Omega\alpha = m^{-1}[\Omega, \Lambda]\alpha = \alpha,$$

as desired (here we have used the fact that Ω preserves the $[\Omega, \Lambda]$ eigenspaces). If, on the other hand, $m = 0$, then $(\mathrm{id} - ip)\alpha = 0$ but $[\Omega, h]$ vanishes as well, by $\Omega\alpha = 0$. □

With this local description of the cohomology of Ω we can now prove Ševera's identification of $H(\mathrm{dR}^{\hbar}_M, \Omega)$ with the sheaf of half-densities on M, following Khudaverdian and Voronov (see Lemma 2.1 of [12]).

Proof of Proposition 8. We define a map of sheaves

$$\psi : H(\mathrm{dR}^{\hbar}_M, \Omega) \to |\Lambda^{\hbar}_M|^{1/2}$$

that induces isomorphisms on stalks as follows. Let $V \subset M$ be an open set and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of V by oriented Darboux charts. Write $\{x^j_i, x^+_{i,j}\}_{j=1,\ldots,n}$ for the coordinates on U_i.

Let $[\alpha] \in H(\mathrm{dR}^{\hbar}_M, \Omega)(V)$ be a cohomology class represented by the form $\alpha \in \mathrm{dR}^{\hbar}_M(V)$. The restrictions $[\alpha]|_{U_i}$ can be represented by the restrictions $\alpha|_{U_i}$, each of which can be written as

$$\alpha|_{U_i} = f_i \, dx_i^1 \cdots dx_i^n,$$

by Lemma 13. Let us write $|\mathcal{D}(x_i, x_i^+)|^{1/2}$ for the frame of the line bundle of half-densities determined by the coordinates on U_i. We claim that the half-densities $f_i \, |\mathcal{D}(x_i, x_i^+)|^{1/2}$ on U_i glue to a half-density on V, giving us the definition of $\psi([\alpha])$.

It is enough to check that the expression $dx^1 \cdots dx^n$ transforms, in the cohomology of Ω, as a half-density. To see this, we first note that for distinct indices i and i'

$$dx_{i'}^j = dx_i^k \frac{\partial x_{i'}^j}{\partial x_i^k} + dx_{i,k}^+ \frac{\partial x_{i'}^j}{\partial x_{i,k}^+},$$

and hence

$$dx_{i'}^1 \cdots dx_{i'}^n = dx_i^1 \cdots dx_i^n \det\left(\frac{\partial x_{i'}}{\partial x_i}\right) + \text{terms containing } dx_i^+. \tag{2.6}$$

The terms containing dx_i^+ have non-zero eigenvalue under $[\Omega, \Lambda]$ and are Ω-closed, whence they are Ω-exact and and vanish in the cohomology of Ω. If T is the matrix of derivatives associated to the coordinate change and P is its upper-left even-even block, then the matrix $\partial x_{i'}/\partial x_i$ appearing in the determinant above is P^{-1}. This determinant $\det P^{-1}$ is positive because x_i and $x_{i'}$ were chosen as positively-oriented charts for U_i and $U_{i'}$. Half-densities, on the other hand, transform as $|\operatorname{Ber}^{1/2}(T)|^{-1}$. Applying Proposition 5 with the choice of an even-odd Lagrangian polarization, we see that

$$|\operatorname{Ber}^{1/2}(T)|^{-1} = |\det(P)|^{-1} = \det(\partial x_{i'}/\partial x_i).$$

Hence $dx^1 \cdots dx^n$ transforms as a half-density and we conclude that ψ is well-defined. A straightforward argument shows that ψ is independent of the choice of (oriented) Darboux cover for V. It follows that ψ commutes with restriction maps and thus defines a map of sheaves. The induced map ψ_p on stalks at any $p \in M$ is clearly an isomorphism.

The map ψ identifies, locally, $f \, dx^1 \cdots dx^n$ with $f \, |\mathcal{D}(x, x^+)|^{1/2}$ and so by the grading conventions of Remark 9, we see that ψ has degree zero and even parity. □

This isomorphism provides a homological characterization of half-densities on odd symplectic supermanifolds. The local expression $dx^1 \cdots dx^n$ for the generator of the cohomology of Ω should not be surprising in light of Proposition 10. Indeed, if a half-density on M restricts to a density on the even Lagrangian defined on U by $\{x_1^+ = \cdots = x_n^+ = 0\}$, it can be identified, using the orientation on the body $|M|$ of M, with the volume form $dx^1 \cdots dx^n$. It is natural to ask, then, how these constructions generalize in the setting of Lagrangians that are not purely even. We postpone addressing this question to Chapter 5. For now we turn to the main focus of this thesis, which is the construction of Khudaverdian's BV operator on half-densities using Proposition 8 and the tools of homological perturbation theory.

CHAPTER 3

Local perturbation theory

We review the constructions of homological perturbation theory that will be useful to us below. Given a (co)chain complex, the machinery of homological perturbation theory arose historically from the problem of finding a smaller, homotopy equivalent complex. We need here only a small piece of the theory: the homological perturbation lemma. Given a perturbation of the differential on the larger complex, the lemma transfers this deformation to the smaller complex, retaining the equivalence. The perturbation lemma was originally studied in the context of the homology of fibrations and the twisted Eilenberg-Zilber theorem [18][6][10], but has since found many applications and extensions outside of algebraic topology. In this thesis we follow [8].

Definition 14. A *strong deformation retraction* of complexes is a diagram

$$(C, d_C) \underset{p}{\overset{i}{\rightleftarrows}} (D, d_D) \circlearrowleft h$$

where i and p are maps of complexes and h is a map of degree -1, such that

$$pi = \mathrm{id}_C, \qquad \mathrm{id}_D - ip = [d_D, h],$$

together with the *side conditions*

$$hi = 0, \qquad ph = 0, \qquad h^2 = 0.$$

Theorem 15 (Homological perturbation lemma)**.** Suppose we have a strong deformation retraction as above, with δ a perturbation of the differential on D. That is, $\delta^2 = 0$ and $(d_D + \delta)^2 = 0$. Suppose moreover that the perturbation is *small*, i.e. the degree zero operator

$(1 - \delta h)$ is invertible. Then, if we write

$$A = (1 - \delta h)^{-1}\delta,$$

there exists a perturbed strong deformation retraction

$$(C, d'_C) \underset{p'}{\overset{i'}{\rightleftarrows}} (D, d'_D = d_D + \delta) \circlearrowleft h'$$

where

$$\begin{aligned} i' &= (1 + hA)i = (1 - hd)^{-1}i \\ p' &= p(1 + Ah) = p(1 - dh)^{-1} \\ h' &= h + hAh \\ d'_C &= d_C + pAi. \end{aligned}$$

The proof of Theorem 15 is a straightforward series of computations and can be found, for instance, in [8]. We remark that the side conditions on the unperturbed data ensure that we still have $p'i' = \mathrm{id}_C$ in the perturbed data.

The notion of a strong deformation retraction systematizes the data used to prove Lemma 13, as the following result shows.

Proposition 16. The inclusion i of the $m = 0$ eigenspace of $[\Omega, \Lambda]$ fits into a strong deformation retract,

$$(\mathrm{dR}^{\hbar}_M(U)_0, 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{dR}^{\hbar}_M(U), \Omega) \circlearrowleft h$$

Proof. We showed in the previous chapter that i and p are maps of complexes and moreover that $\mathrm{id} - ip = [\Omega, h]$. It remains to check that the side conditions hold. The first two side conditions $hi = 0$ and $p\Lambda = 0$ follow simply from the fact that Λ kills $dx^1 \cdots dx^n$. The third side condition $h^2 = 0$ is immediate because $|\Lambda| = 1$ and thus $\Lambda^2 = 0$. $\square$

We will now obtain the BV Laplacian on U via a perturbation of the differential Ω on the right-hand-side of this strong deformation retraction. In particular, we reintroduce the de Rham differential $d = d_{\mathrm{dR}}$.

Lemma 17. The de Rham differential d is a small perturbation of the differential Ω on $(\mathrm{dR}^{\hbar}_M(U), \Omega)$.

Proof. The two-form Ω is symplectic and in particular closed, so

$$(\Omega + d)^2\alpha = \left(\hbar^{-2}\omega^2 + \hbar^{-1}\omega d + \hbar^{-1} d\omega + d^2\right)$$
$$= \hbar^{-1}(\omega d\alpha + d\omega\alpha) = 0.$$

To check that d is a small perturbation we consider

$$(1 - dh)^{-1} = \sum_{i=1}^{\infty}(dh)^i$$

and recall that while d increases de Rham degree by 1, the homotopy operator h is proportional to Λ, which decreases de Rham degree by 2. Hence the sum is finite when applied to any de Rham form. □

Now, as in the statement of the homological perturbation lemma, define

$$A = (1 - dh)^{-1}d = d + dhd + dhdhd + \cdots .$$

Applying Theorem 15 to the strong deformation retraction above with the perturbation $\delta = d$, we obtain on the left the transferred differential

$$pAi = p(d + dhd + dhdhd + \cdots)i.$$

This sum simplifies significantly, as the following lemma shows.

Lemma 18. Every term in the sum pAi above vanishes except the second:

$$pAi = p\sum_{j=1}^{\infty}(dh)^j di = pdhdi.$$

Proof. We argue by de Rham degree counting: i produces forms of de Rham degree n and p vanishes on forms of de Rham degree different than n. The operator $(dh)^j d$ has de Rham degree $1 - j$, so $(dh)^j di$ has de Rham degree n only when $j = 1$. Thus the only term that survives the application of p is the $j = 1$ term. □

Theorem 19. Transferring the perturbation $d = d_{\mathrm{dR}}$ of the strong deformation retraction in Proposition 16 yields the BV Laplacian

$$pAi = \hbar\,\Delta$$

on the $m = 0$ eigenspace of $[\Omega, \Lambda]$. The resulting strong deformation retraction is

$$(\mathrm{dR}^{\hbar}_M(U)_0, \hbar\,\Delta) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathrm{dR}^{\hbar}_M(U), \Omega + d) \circlearrowleft h'$$

where

$$i' = (1 + hA)i \qquad p' = p(1 + Ah) \qquad h' = h + hAh$$

Proof. Apply the homological perturbation lemma (Theorem 15) and Lemma 18 to obtain the differential $pAi = pdhdi$ on the left. Suppose now that $\alpha = f(x, x^+)dx^1 \cdots dx^n \in \mathrm{dR}^{\hbar}_M(U)_0$. Then

$$\begin{aligned}
pAi\alpha &= pdhd\alpha \\
&= pdh\left(dx^+_k \frac{\partial f}{\partial x^+_k} dx^1 \cdots dx^n\right) \\
&= pd\Lambda\left(dx^+_k \frac{\partial f}{\partial x^+_k} dx^1 \cdots dx^n\right) \\
&= pd\left(\hbar\iota(\partial_{x^j})\delta^j_k \frac{\partial f}{\partial x^+_k} dx^1 \cdots dx^n\right) \\
&= (-1)^{|f|+k}\hbar pd\left(\frac{\partial f}{\partial x^+_k} dx^1 \cdots \widehat{dx^k} \cdots dx^n\right).
\end{aligned}$$

The de Rham differential now introduces two types of terms: one of the form $dx_j^+ \partial^2 f / \partial x_j^+ \partial x_i^+$ and one of the form $dx^j \partial^2 f / \partial x^j \partial x_k^+$. The eventual application of p (which is nonzero only on monomials of the form $dx^1 \cdots dx^n$) allows us to drop the terms with dx_j^+. Thus we are left with

$$\begin{aligned} pAi\alpha &= (-1)^{|f|+k} \hbar p \left(dx^j \frac{\partial^2 f}{\partial x^j \partial x_k^+} dx^1 \cdots \widehat{dx^k} \cdots dx^n \right) \\ &= \hbar \frac{\partial^2 f}{\partial x^k \partial x_k^+} dx^1 \cdots dx^n \\ &= \hbar \Delta \alpha, \end{aligned}$$

the BV Laplacian, as desired. □

The resulting strong deformation retraction gives us a map i' that intertwines the BV Laplacian with $\Omega + d$:

$$i'(\hbar \Delta \alpha) = (\Omega + d) i'(\alpha). \tag{3.1}$$

This gives us a formula that reduces the second-order differential operator Δ to the first-order differential operator $\Omega + d$. A straightforward inductive computation reveals the general formula for i'.

Lemma 20. Let $\alpha = f(x, x^+)\, dx^1 \cdots dx^n \in \mathrm{dR}_M^\hbar(U)_0$. Then

$$\begin{aligned} i'(\alpha) = \sum_{j=0}^{n} \sum_{k_1 < \cdots < k_j} & (-1)^{j|f| + k_1 + \cdots + k_j} \cdot \\ & \cdot \hbar^j \frac{\partial^j f}{\partial x_{k_j}^+ \cdots \partial x_{k_1}^+} dx^1 \cdots \widehat{dx^{k_1}} \cdots \widehat{dx^{k_j}} \cdots dx^n. \end{aligned}$$

We end this chapter by considering the case of a differential graded supermanifold M, with differential arising in physics from the choice of an action. We consider, in particular, a function $S_0 \in \mathcal{O}_M$ of even internal parity and ghost number zero satisfying the *classical master equation*

$$(S_0, S_0) = 0.$$

As the name suggests, the classical master equation is obtained from the quantum master equation upon specializing to $\hbar = 0$. We have written S_0 for this function because the solution S_{BV} to the quantum master equation, Equation 1.1, is typically constructed order-by-order in $\hbar$ as a power series with $S_{\text{BV}} = S_0 + O(\hbar)$.

The pairing $(-,-)$ is the antibracket: the Poisson bracket associated to the symplectic form ω. The antibracket is defined by $(g, g') = X_g(g')$ where X_g is the Hamiltonian vector field for g under ω. Alternatively it can be defined (in our context of eigenvalue $m = 0$ forms) as follows: multiplication $m((g, g'))$ by (g, g') is given by

$$m((g, g')) = (-1)^{|g|}[[\Delta, g], g']. \tag{3.2}$$

That is, the two-fold commutator of the second-order differential operator Δ with functions on M is the zeroth-order differential operator defining the antibracket. The Jacobi identity for the antibracket ensures that the differential

$$s = (S_0, -) : \mathcal{O}_M^{\hbar} \to \mathcal{O}_M^{\hbar}$$

of ghost number 1 squares to zero, $s^2 = 0$.

We may view the derivation s as the Lie derivative $s = L_Q$ along the Hamiltonian vector field Q of S_0. The vector field Q has ghost number 1 on M and even internal parity. This Lie derivative extends as usual to an operator on the de Rham complex $\mathrm{dR}_M^{\hbar}$ given by the Cartan homotopy formula

$$s = L_Q = [\iota(Q), d] = \iota(Q)d - d\iota(Q).$$

Notice that the graded commutator here is the usual commutator as $\iota(Q)$ is even: contraction decreases de Rham degree by 1 while Q increases ghost number by 1. We place the contraction first in order to keep $L_Q f = Q(f)$.

Lemma 21. The operator $s = L_Q$ is a differential on $\mathrm{dR}_M^{\hbar}$ that commutes with Ω and the de Rham differential d.

Proof. The assignment of a Hamiltonian vector field X_f to a function f maps the antibracket to the commutator of vector fields. Hence the classical master equation $(S_0, S_0) = 0$ implies that $[Q, Q] = 0$. It follows that

$$2L_Q^2 = [L_Q, L_Q] = L_{[Q,Q]} = 0$$

and hence $s = L_Q$ is a differential. As Q is a Hamiltonian vector field, we have that

$$L_Q\omega = \iota(Q)d\omega - d\iota(Q)\omega = -ddS_0 = 0,$$

from which we find

$$\begin{aligned}[L_Q, \Omega]\alpha &= \hbar^{-1}L_Q\omega \wedge \alpha + \hbar^{-1}\omega L_Q\alpha \\ &= \hbar^{-1}(L_Q\omega)\alpha - \hbar^{-1}\omega L_Q\alpha + \hbar^{-1}\omega L_Q\alpha \\ &= 0.\end{aligned}$$ □

The previous lemma demonstrates that $d + s$ is a perturbation of Ω on the de Rham complex and we can now transfer this perturbation to the $m = 0$ eigenspace of $[\Omega, \Lambda]$.

Proposition 22. Transferring the perturbation $d + s = d_{\mathrm{dR}} + (S_0, -)$ of the strong deformation retraction in Proposition 16 yields the differential

$$pAi = \hbar\Delta + (S_0, -) + \Delta S_0$$

on the $m = 0$ eigenspace of $[\Omega, \Lambda]$. The resulting strong deformation retraction is

$$(\mathrm{dR}^\hbar_M(U)_0, \hbar\Delta + (S_0, -) + \Delta S_0) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathrm{dR}^\hbar_M(U), \Omega + d + s) \circlearrowleft h'$$

with the notation of Theorem 19, where $A = (1 - (d + s)h)^{-1}(d + s)$.

Proof. First we note that $d + s$ is a small perturbation of Ω:

$$(1 - (d + s)h)^{-1} = \sum_{k=1}^{\infty}(dh + sh)^k$$

is a finite sum because $dh + sh$ reduces de Rham degree by at least 1 and is therefore nilpotent. Applying the homological perturbation lemma, Theorem 15, we obtain a new differential on the left,

$$pAi = p(1 - (d + s)h)^{-1}(d + s)i$$

As in the proof of Theorem 19, we can argue by de Rham degree to compute. For $\alpha = f(x,x^+)dx^1\cdots dx^n \in \mathrm{dR}^{\hbar}_M(U)_0$,

$$\begin{aligned} pAi\alpha &= p\left(1 + (d+s)h + (d+s)h(d+s)h + \cdots\right)(d+s)\alpha \\ &= ps\alpha + pdhd\alpha. \end{aligned}$$

As before, the second term is computed to be $pdhd\alpha = \hbar\Delta\alpha$, while the first term is new,

$$\begin{aligned} ps\alpha &= pL_Q\left(f(x,x^+)dx^1\cdots dx^n\right) \\ &= (S_0, f)dx^1\cdots dx^n + (-1)^{|f|} f pL_Q dx^1\cdots dx^n. \end{aligned}$$

A short computation reveals that

$$Q = \frac{\partial S_0}{\partial x_k^+}\frac{\partial}{\partial x^k} + \frac{\partial S_0}{\partial x^k}\frac{\partial}{\partial x_k^+},$$

from which we find

$$pL_Q dx^1\cdots dx^n = \frac{\partial^2 S_0}{\partial x^k \partial x_k^+} dx^1\cdots dx^n.$$

Thus

$$ps\alpha = \left((S_0, f) + \frac{\partial^2 S_0}{\partial x^k \partial x_k^+} f\right) dx^1\cdots dx^n$$

and the result follows. □

In the next chapter we will identify the $m = 0$ eigenspace of $[\Omega, \Lambda]$ on U with half-densities on U. With this in mind, we may interpret the transferred differential

$$((S_0, -) + \Delta S_0) f dx^1\cdots dx^n = \left((S_0, f) + \frac{\partial^2 S_0}{\partial x^k \partial x_k^+} f\right) dx^1\cdots dx^n$$

as the lift of the Hamiltonian vector field Q to the space of half-densities. Such a lift is constructed in Section 2 of [9] as the first-order differential operator

$$\mathbf{H}_g = (-1)^{|g|}[\Delta, m(g)],$$

where $m(g)$ denotes multiplication by a function g.

Proposition 23. The transferred perturbation of Proposition 22 can be written

$$(S_0, -) + \Delta S_0 = \mathbf{H}_{S_0}$$

Proof. We calculate using Equation 3.2:

$$\begin{aligned}((S_0, f) + \Delta S_0 f)\, dx^1 \cdots dx^n &= ([[\Delta, S_0], f] + \Delta S_0 f)\, dx^1 \cdots dx^n \\ &= \Big([\Delta, S_0] f - (-1)^{|f|} f [\Delta, S_0] + \Delta S_0 f\Big)\, dx^1 \cdots dx^n \\ &= \mathbf{H}_{S_0} f\, dx^1 \cdots dx^n.\end{aligned}$$

□

CHAPTER 4

Čech globalization

In the previous chapter we constructed the BV Laplacian Δ locally, in an oriented Darboux coordinate chart U on the 0-eigenspace $\mathrm{dR}^{\hbar}_M(U)_{m=0}$ of $[\Omega, \Lambda]$. Recall that, by Proposition 16, the inclusion of this eigenspace into the de Rham complex $\mathrm{dR}^{\hbar}_M(U)$ induces an isomorphism on the cohomology of Ω. In line with the identification ψ of Proposition 8 between half-densities on M and the cohomology of Ω, we will identify a half-density $\mu = f\,|\mathcal{D}(x, x^+)|^{1/2} \in |\Lambda^{\hbar}_M|^{1/2}(U)$ on an oriented Darboux chart U with the form $f\,dx^1 \cdots dx^n \in \mathrm{dR}^{\hbar}_M(U)_{m=0}$. This gives us an isomorphism

$$|\Lambda^{\hbar}_M|^{1/2}(U) \cong \mathrm{dR}^{\hbar}_M(U)_{m=0}$$

of rank 1 $\mathcal{O}^{\hbar}_M(U)$-modules. Thus we may replace the left-hand side of the strong deformation retraction in Theorem 19 with the space of half-densities on U:

$$(4.1) \qquad \left(|\Lambda^{\hbar}_M|^{1/2}(U), \hbar\,\Delta\right) \underset{p'}{\overset{i'}{\rightleftarrows}} \left(\mathrm{dR}^{\hbar}_M(U), \Omega + d\right) \circlearrowleft h'$$

where

$$\hbar\,\Delta f |\mathcal{D}(x, x^+)|^{1/2} = \hbar\,\frac{\partial^2 f}{\partial x^i \partial x^+_i} |\mathcal{D}(x, x^+)|^{1/2}.$$

Extending the BV Laplacian to a global differential on the sheaf of half-densities takes more work, however, as the formulas above are coordinate-dependent. Equation 2.6, for instance, demonstrates that $dx^1 \cdots dx^n$ transforms as a half-density only after passing to cohomology. As a consequence, the map i of Proposition 16 does not globalize to a map on the sheaf of half-densities. In this chapter we approach the problem of globalization by combining homological perturbation theory with Čech methods. Our goal is to prove the following:

Theorem 24. The local expression $\hbar\, \partial^2/\partial x^i \partial x_i^+$ for the BV Laplacian in Equation 4.1 globalizes to a differential on the sheaf of half-densities $|\Lambda_M^\hbar|^{1/2}$.

Fix an atlas $\mathcal{U} = \{U_i\}_{i \in I}$ of M consisting of oriented Darboux charts. For $V \subset M$ an open set we write, abusing notation slightly, $\mathcal{U}$ for the open cover $\mathcal{U} \cap V$. We construct the BV Laplacian Δ over each open set V in such a way that on an oriented Darboux chart the construction reduces to the one above.

We might start by naively upgrading the map i to a map between the Čech total complexes of half-densities (with differential $\check{d}$) and de Rham forms (with differential $\Omega + \check{d}$). This approach immediately runs into a problem: such a map i does not intertwine with the differentials, $(\Omega + \check{d})i = \check{d}i \neq i\check{d}$. The reason for this is straightforward — half-densities transform as the square root of the Berezinian but the expression $dx^1 \cdots dx^n$ does not, according to Equation 2.6.

The map i needs to modified for it to be a map of complexes. We use homological perturbation theory to construct this modification. To start, we set up a strong deformation retraction between the Čech total complexes of half-densities and de Rham forms, where the differential on the left is 0 and the differential on the right is Ω, dropping the Čech differential for now. In total degree k, the total complexes of these Čech complexes are given

$$(\mathrm{Tot}^k\, \check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2}), 0) = \prod_{p+q=k} \prod_{i_0,\ldots,i_p} (|\Lambda_M^\hbar|^{1/2})^q (U_{i_0 \cdots i_p})$$

and

$$(\mathrm{Tot}^k\, \check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar), \Omega) = \prod_{p+q=k} \prod_{i_0,\ldots,i_p} (\mathrm{dR}_M^\hbar)^q (U_{i_0 \cdots i_p}),$$

As before, the grading on $|\Lambda_M^\hbar|^{1/2}$ is by ghost number, while the grading on $\mathrm{dR}_M^\hbar$ is the sum of ghost number and de Rham degree.

Define an extension of the map i from the local setting to the Čech setting,

$$i : (\mathrm{Tot}^*\, \check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2}), 0) \to (\mathrm{Tot}^*\, \check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar), \Omega),$$

as follows. If $U_i \in \mathcal{U}$, denote by $\{x_i^j, x_{i,j}^+\}_{j=1,\ldots,n}$ the corresponding Darboux coordinates. Write

$$|\mathcal{D}(x_i, x_i^+)|^{1/2} \in |\Lambda_M^\hbar|^{1/2}(U_i)$$

for the basis half-density determined by these coordinates. Suppose $\mu \in \mathrm{Tot}^k\, \check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2})$ is a k-cochain valued in half-densities, written

$$\mu = (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \dots)$$

where $\mu^{(j)}$ has Čech degree j and ghost number $k-j$. On the intersection $U_{i_0 \dots i_j} = U_{i_0} \cap \dots \cap U_{i_j}$ we have available $j+1$ different coordinate systems. We choose the coordinates on the first open set U_{i_0}. We write, in particular, the half-density $\mu^{(j)}_{i_0 \cdots i_j}$ in terms of the coordinate half-density on U_{i_0} as

$$\mu^{(j)}_{i_0 \cdots i_j} = f_{i_0 \cdots i_j}\, |\mathcal{D}(x_{i_0}, x^+_{i_j})|^{1/2},$$

for some function $f_{i_0 \cdots i_j} \in \mathcal{O}_M^\hbar(U_{i_0 \cdots i_j})$. With this notation fixed, the map i is now defined

$$i(\mu)^{(j)}_{i_0 \cdots i_j} = f_{i_0 \cdots i_j}\, dx^1_{i_0} \cdots dx^n_{i_0},$$

This is a map of complexes as

$$\Omega\, i(\mu) = 0$$

(recall that the differential on the source is taken to be zero).

Next we construct the maps p and h required for a strong deformation retraction. As before, these maps rely on Lemma 12. Hence we first define the map Λ. For $\alpha \in \mathrm{Tot}^k\, \check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar)$ written as $\alpha = (\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \dots)$, we define

$$\Lambda(\alpha)^{(j)}_{i_0 \cdots i_j} = \hbar\, \iota(\partial_{x^\ell_{i_0}})\iota(\partial_{x^+_{i_0,\ell}})\alpha^{(j)}_{i_0 \cdots i_j},$$

where, again, we have chosen to use the coordinates from the first open set U_{i_0}. By Lemma 12, the de Rham complex over each intersection $U_{i_0 \cdots i_j}$ splits

$$\mathrm{dR}_M^\hbar(U_{i_0 \cdots i_j}) = \bigoplus_{m=0} \mathrm{dR}_M^\hbar(U_{i_0 \cdots i_j})_m$$

along the eigenspaces of the commutator $[\Omega, \Lambda]$.

$$h = \begin{cases} 0 & m = 0, \\ m^{-1}\Lambda & m > 0. \end{cases}$$

More explicitly, write

$$\alpha^{(j)}_{i_0\cdots i_j} = \sum_{m=0}^{\infty} \alpha^{(j),m}_{i_0\cdots i_j},$$

for the local splitting along $[\Omega, \Lambda]$ (the sum is finite for each j). Then

$$(h\alpha)^{(j)}_{i_0\cdots i_j} = \sum_{m=1}^{\infty} m^{-1} \Lambda \alpha^{(j),m}_{i_0\cdots i_j}.$$

In the same notation, the map p is given by the composition

$$\begin{aligned} \sum_{m=0}^{\infty} \alpha^{(j),m}_{i_0\cdots i_j} \mapsto \alpha^{(j),m=0}_{i_0\cdots i_j} &= f_{i_0\cdots i_j}\, dx^1_{i_0} \cdots dx^n_{i_0} \\ \mapsto p(\alpha)^{(j)}_{i_0\cdots i_j} &= f_{i_0\cdots i_j}\, |\mathcal{D}(x_{i_0}, x^+_{i_0})|^{1/2}. \end{aligned}$$

We note that p is a map of complexes: p applied to the image of Ω is zero as Ω kills the kernel of $[\Omega, \Lambda]$.

Proposition 25. With i, p, and h defined as above, there is a strong deformation retraction

$$(\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda^\hbar_M|^{1/2}), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathrm{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}^\hbar_M), \Omega) \circlearrowleft h$$

Proof. The proof is essentially the same as the local case: that h is a homotopy is a straightforward computation, and the side conditions

$$hi = 0, \qquad ph = 0, \qquad h^2 = 0,$$

are similarly easy to check. □

We can now reintroduce the Čech differential as a perturbation of the differential Ω on the Čech-de Rham side, apply the perturbation lemma, and study the resulting transferred differential. First a small point about signs: the ungraded commutator of Ω and $\check{d}$ is zero, so we replace Ω by $(-1)^p \Omega$ before perturbation, where p is the Čech degree. To preserve the strong deformation retraction above we scale Λ, and hence h, by $(-1)^p$ as well.

Lemma 26. The perturbation $\check{d}$ of $(-1)^p\Omega$ on the Čech-de Rham complex is small; that is, $(1 - \check{d} \cdot (-1)^p h)$ is invertible.

Proof. The inverse is given by the sum

$$(1 - \check{d} \cdot (-1)^p h)^{-1} = \sum_{i=1}^{\infty} (\check{d} \cdot (-1)^p h)^i,$$

which, we note, is not a finite sum: unlike in the case of the perturbation by d, it is not enough to note that h decreases de Rham degree by 2. Indeed, the Čech-de Rham complex is an infinite product and thus may have cochains of increasing de Rham degree (in a fixed total degree).

The sum is, however, finite on each factor of this infinite product, and is thus well-defined. If we write a Čech-de Rham cochain α_0 in terms of its Čech components as before,

$$\alpha_0 = (\alpha_0^{(0)}, \alpha_0^{(1)}, \ldots),$$

and write

$$\alpha_i = (h\check{d})^{i-1} h\alpha_0,$$

then we find that

$$\begin{aligned}\sum_{i=1}^{\infty} (\check{d} \cdot (-1)^p h)^i = \alpha_0 &\pm (0, \check{d}\alpha_1^{(0)}, \check{d}\alpha_1^{(1)}, \ldots) \\ &\pm (0, 0, \check{d}\alpha_2^{(0)}, \check{d}\alpha_2^{(1)}, \ldots) \\ &\pm \ldots\end{aligned}$$

That the sum is finite in each factor is now apparent. □

Hence the perturbation $\check{d}$ of Ω on the Čech-de Rham complex can be transferred to the Čech complex of half-densities. The following shows that the resulting perturbation on the left is the Čech differential.

Proposition 27. Applying the homological perturbation lemma yields the strong deformation retract

$$(\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2}), \check{d}) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathrm{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar), (-1)^p\Omega + \check{d}) \circlearrowleft h'$$

In particular, the perturbation $\check{d}$ on the right is transferred to the Čech differential on the left in such a way that i' and p' are maps of complexes. Explicitly,

$$i' = (1 - (-1)^p h\check{d})^{-1} i, \qquad p' = p(1 - \check{d} \cdot (-1)^p h)^{-1},$$

and

$$h' = (-1)^p h + hAh.$$

Proof. The new differential on the left is

$$p(1 - \check{d}(-1)^p h)^{-1}\check{d}i = p\check{d}i + p\check{d} \cdot (-1)^p h\check{d}i + \cdots$$

Recall that p is nonzero only on forms of de Rham degree n. Any application of h reduces de Rham degree by 2 (and the Čech differential, of course, does not change the de Rham degree). Hence each term but the first in this sum vanishes.

The first term is precisely the Čech differential for the Čech complex of $|\Lambda_M^\hbar|^{1/2}$. This is a simple consequence of the transformation properties of the form $dx^1 \cdots dx^n$, but we write out the calculation explicitly. Suppose $\mu = (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \ldots)$ is a (homogeneous) Čech cochain on the left. Then

$$(\check{d}_{\mathrm{dR}} i\mu)^{(j+1)}_{i_0 \cdots i_{j+1}} = (i\mu)^{(j)}_{i_1 \cdots i_{j+1}}\Big|_{U_{i_0 \cdots i_{j+1}}} + \sum_{k=1}^{j+1} (-1)^k (i\mu)^{(j)}_{i_0 \cdots \widehat{i_k} \cdots i_{j+1}}\Big|_{U_{i_0 \cdots i_{j+1}}}.$$

The first term is written as, in the notation from above, $f_{i_1 \cdots i_{j+1}}\, dx^1_{i_1} \cdots dx^n_{i_1}$. Applying p to the last j terms clearly yields the last j terms of the Čech differential on half-densities — only the first term requires analysis. From Equation 2.6 we know that

$$dx_{i_1}^1 \cdots dx_{i_1}^n = |\operatorname{Ber}^{1/2}(T)|^{-1} dx_{i_0} \cdots dx_{i_0}^n + \Omega(\cdots)$$

(where T is the matrix of derivatives associated to the coordinate change) and so applying p on $U_{i_0 \cdots i_{j+1}}$ to the first term yields

$$f_{i_1 \cdots i_{j+1}} |\operatorname{Ber}^{1/2}(T)|^{-1} |\mathcal{D}(x_{i_0}, x_{i_0}^+)|^{1/2} = f_{i_1 \cdots i_{j+1}} |\mathcal{D}(x_{i_1}, x_{i_1}^+)|^{1/2}.$$

This is precisely the first term of the Čech differential on half-densities. □

The perturbation lemma provides us with a modified map i' that intertwines the differentials $\check{d}$ and $(-1)^p\Omega + \check{d}$. This puts us in a position to imitate the local perturbation argument from the previous chapter. We perturb the Čech-de Rham complex by the de Rham differential $d = d_{\mathrm{dR}}$ and transfer the perturbation. Again, as is standard for total complexes, we will use $(-1)^p d$ where p is the Čech degree. The proof that the perturbation is small is identical to the proof of Lemma 26.

Lemma 28. The perturbation by the de Rham differential $(-1)^p d$ of $(-1)^p\Omega + \check{d}$ on the Čech-de Rham complex is small.

Proposition 29. Applying the homological perturbation lemma yields the strong deformation retract

$$(\operatorname{Tot}^* \check{C}(\mathcal{U}, |\Lambda_M^\hbar|^{1/2}), \check{d} + p'A'i') \underset{p''}{\overset{i''}{\rightleftarrows}} (\operatorname{Tot}^* \check{C}(\mathcal{U}, \mathrm{dR}_M^\hbar), (-1)^p(\Omega + d) + \check{d}) \circlearrowleft h''$$

where the definitions of i'', p'', and h'' are as usual. The transferred perturbation is locally just the BV Laplacian.

Proof. We temporarily drop the signs $(-1)^p$ for clarity as they do not affect the calculation. We will reintroduce them below when necessary. We consider the new differential on the left

$$\check{d} + p'A'i' = \check{d} + (p + pAh)\sum_{j=0}^{\infty}(dh')^j d(i + hAi),$$

where

$$A = (1 - \check{d}h)^{-1}\check{d} \qquad \text{and} \qquad A' = (1 - dh')^{-1}d.$$

Expanding the two binomials present in the perturbing term, let us consider the resulting four sums separately.

The first term is

$$p\sum_{j=0}^{\infty}(dh')^j di = p\sum_{j=0}^{\infty}(dh + dhAh)^j di$$

If we again keep in mind that i produces forms of de Rham degree n, p kills forms of de Rham degree different than n, and h reduces de Rham degree by 2, we see that the only nonvanishing terms are those with twice as many d's present as h's. Thus we are left with $pdhdi$ (part of the $j = 1$ term).

The remaining three sums are:

$$p\sum_{j=0}^{\infty}(dh')^j dhAi = p\sum_{j=0}^{\infty}(dh + dhAh)^j dhAi$$

$$pAh\sum_{j=0}^{\infty}(dh')^j di = pAh\sum_{j=0}^{\infty}(dh + dhAh)^j di$$

$$pAh\sum_{j=0}^{\infty}(dh')^j dhAi = pAh\sum_{j=0}^{\infty}(dh + dhAh)^j dhAi.$$

Recalling that $A = \check{d} + \check{d}h\check{d} + \cdots$, we see that there are no terms in any of these sums that contain twice as many occurrences as the de Rham differential as the homotopy operator h. Hence all three vanish for de Rham degree reasons.

In total, then, we obtain the differential

$$\check{d} + p'A'i' = \check{d} + (-1)^p pdhdi$$

on the Čech complex of half-densities, where we have reintroduced the signs $(-1)^p$. This is precisely the local expression for the BV Laplacian. In Čech degree zero, in particular, and on an oriented Darboux coordinate chart U_{i_0}, the local perturbation theory calculation of Theorem 19 yields

$$f|\mathcal{D}(x_{i_0}, x^+_{i_0})|^{1/2} \mapsto \hbar \frac{\partial^2 f}{\partial x^j_{i_0} \partial x^+_{i_0,j}} |\mathcal{D}(x_{i_0}, x^+_{i_0})|^{1/2}. \qquad \square$$

The new differential on the left is the sum $\check{d} + (-1)^p pdhdi$. As a differential, this sum squares to zero, and in particular we see that $\check{d}$ commutes with the transferred perturbation $(-1)^p pdhdi$. In particular, the transferred perturbation has Čech degree zero, and hence sends Čech zero-cocycles to Čech zero-cocycles. Restricting attention to the Čech zero-cocycles, then, we arrive at a new proof of Khudaverdian's Theorem 1, independent of Khudaverdian's proof, that Δ defines an operator acting globally on half-densities.

Proof of Theorem 24. Let $V \subset M$ be an open set and $\mu \in |\Lambda^\hbar_M|^{1/2}(V)$ be a half-density defined on V. Abusing notation slightly, we write $\mathcal{U} = \{U_i\}_{i\in I}$ for the cover of V induced by the oriented Darboux atlas for M. We use the same notation as before: $\mu|_{U_i} = f_i\,|\mathcal{D}(x_i, x^+_i)|^{1/2}$. The inclusion $\sqcup U_i \hookrightarrow V$ induces a map $|\Lambda^\hbar_M|^{1/2}(V) \longrightarrow \check{C}^*(\mathcal{U}, |\Lambda^\hbar_M|^{1/2})$ that sends μ to the Čech 0-cocycle $\mu = (\mu^{(0)}, 0, 0, \ldots)$. As the differential $\check{d} + (-1)^p pdhdi$ squares to zero, we have that

$$\check{d}pdhdi - pdhdi\check{d} = 0.$$

Applying this differential to the 0-cocycle μ, we find that

$$(\check{d}pdhdi\mu) = (0, (\check{d}pdhdi\mu)^{(1)}, 0, 0, \ldots) = 0,$$

and thus, for all $i_0, i_1 \in I$,

$$(\check{d}pdhdi\mu)^{(1)}_{i_0 i_1} = (pdhdi\mu)^{(0)}_{i_1} - (pdhdi\mu)^{(0)}_{i_0} = 0$$

on $U_{i_0 i_1}$. Applying the calculations of Theorem 19 we find that

$$\frac{\partial^2 f_{i_1}}{\partial x^j_{i_1} \partial x^+_{i_1,j}} |\mathcal{D}(x_{i_1}, x^+_{i_1})|^{1/2} - \frac{\partial^2 f_{i_0}}{\partial x^j_{i_0} \partial x^+_{i_0,j}} |\mathcal{D}(x_{i_0}, x^+_{i_0})|^{1/2} = 0 \tag{4.2}$$

on $U_{i_0 i_1}$. This is precisely the statement that the local expression for the BV Laplacian yields a global BV Laplacian acting on $|\Lambda^{\hbar}_M|^{1/2}(V)$. In other words, if μ is a Čech 0-cocycle as above, then the BV Laplacian $\check{d} + pdhdi = pdhdi$ applied to μ is again a Čech 0-cocycle.

This argument yields a BV Laplacian Δ on half-densities over each open set $V \subset M$. It is clear that Δ defines an operator on the sheaf of half-densities, as the construction is compatible with restriction maps. □

Let us expand on the proof Theorem 24: we describe more explicitly how the BV Laplacian can be constructed on any given open set V from the result of Proposition 29. The sheaf of half-densities is the sheaf of sections of a smooth line bundle on M, and is hence a fine sheaf. That is, it admits a partition of unity subordinate to any cover. Hence, for an open subset $V \subset M$, the inclusion $\check{\imath}$ of global half-densities as 0-cocycles in the Čech complex for the open cover $\mathcal{U}$ (which is, again, the induced oriented Darboux atlas on V) fits into a (non-canonical) strong deformation retract. Hence the perturbation $(-1)^p pdhdi$ of $\check{d}$ on the right-hand side of the strong deformation retraction in Proposition 29 can be transferred to the BV Laplacian on $\Gamma(V, |\Lambda^{\hbar}_M|^{1/2})$.

Choose a partition of unity ψ_i subordinate to $\mathcal{U}$ and consider the strong deformation retraction

$$(\Gamma(V, |\Lambda^{\hbar}_M|^{1/2}), 0) \underset{\check{p}}{\overset{\check{\imath}}{\rightleftarrows}} (\mathrm{Tot}^* \check{C}(\mathcal{U}, |\Lambda^{\hbar}_M|^{1/2}), \check{d}) \circlearrowleft \check{h}$$

where, for $\mu = (\mu^{(0)}, \mu^{(1)}, \ldots) \in \mathrm{Tot}^k \check{C}(\mathcal{U}, |\Lambda^{\hbar}_M|^{1/2})$,

$$(\check{h}\mu)^{(j-1)}_{i_0 \cdots i_{j-1}} = \sum_{i \in I} \psi_i \mu^{(j)}_{i i_0 \cdots i_{j-1}},$$

and

$$\check{p}\mu = \sum_{i \in I} \psi_i \mu^{(0)}_i.$$

Then a straightforward computation reveals that

$$\mathrm{id} - \check{\imath}\check{p} = [\check{d}, \check{h}],$$

(for details, see for instance, the remarks following Proposition 8.5 of [5]), completing the construction of the strong deformation retraction.[1]

The differential $(-1)^p pdhdi$ is a small perturbation of the differential $\check{d}$ on the right, as the product of $\check{h}$ and $(-1)^p pdhdi$ decreases Čech degree and is thus nilpotent. Thus we obtain

$$\check{A} = (-1)^p pdhdi - pdhdi\check{h}pdhdi + \cdots$$

and a transferred differential $\check{p}\check{A}\check{\imath}$ on the left. Only the first term in the resulting sum survives because $\check{p}$ is nontrivial only on cochains of Čech degree 0. Hence, given $\mu \in \Gamma(V, |\Lambda^{\hbar}_M|^{1/2})$ such that locally on U_i, $\mu|_{U_i} = f_i\,|\mathcal{D}(x_i, x_i^+)|^{1/2}$, we compute

$$\check{p}\check{A}\check{\imath}\mu = \hbar \sum_{i\in I} \psi_i \cdot \frac{\partial^2 f_i}{\partial x_i^j \partial x_{i,j}^+}\,|\mathcal{D}(x_i, x_i^+)|^{1/2}.$$

At any given point $x \in V$ we can rewrite this sum in terms of a single coordinate system $\{x_{i_0}, x_{i_0}^+\}$ (where $U_{i_0} \ni x$) by Equation 4.2 in the proof of Theorem 24. Hence the sum collapses to $\sum_{i\in I} \psi_i = 1$ and we obtain, independent of the choice of partition of unity, Khudaverdian's sheaf of complexes of half-densities — for $V \subset M$ any open set, we have constructed

$$(\Gamma(V, |\Lambda^{\hbar}_M|^{1/2}),\ \hbar\,\Delta).$$

[1]This diagram is, in fact, not a strong deformation retraction as two of the side conditions $\check{p}\check{h} = 0$ and $\check{h}^2 = 0$ do not hold. We can nevertheless apply the homological perturbation lemma to transfer perturbations, though the resulting $\check{\imath}'$ and $\check{p}'$ will not necessarily satisfy $\check{p}'\check{\imath}' = \mathrm{id}$. We note that these two side conditions do indeed hold if we work instead with alternating Čech cochains.

CHAPTER 5

Integral forms

Recall from the local calculations above that we embed half-densities into the complex of de Rham forms as representatives

$$\alpha = f(x, x^+)\, dx^1 \cdots dx^n$$

of the cohomology of Ω. The x^i have even total parity (by definition of oriented Darboux coordinates) and so α may be integrated along the even Lagrangian

$$L = \{x_1^+ = \cdots = x_n^+ = 0\} \subset M.$$

It is natural to ask, then, whether we might instead represent half-densities as forms integrable along Lagrangian $L \subset M$ that are not purely even. We focus in this chapter on the other extreme: we represent half-densities as forms integrable on purely odd Lagrangians $L \subset M$. Throughout this chapter we use the notations of Chapter 4, Section 5.4 of Manin [14].

We leave the case of general, mixed parity Lagrangians to future study, as the theory of pseudodifferential forms of Bernstein and Leites [3] is considerably more subtle than that of differential and integral forms, and our methods do not seem to immediately generalize. The pseudodifferential form

$$\exp\left(-\sum_{i=1}^{n}(dx_i^+)^2\right),$$

for example, does not have a well-defined de Rham degree and is not an eigenvector of $[\Omega, \Lambda]$. One might introduce a grading known as the *picture number* (see, e.g., [2] and [7]) on the class of pseudodifferential forms that are functions of dx^i and dx_i^+ but supported only along $\{dx_i^+ = 0\}_{i=1,\ldots,n}$, such as

$$f\, dx^1 \cdots dx^n \delta(dx_1^+) \cdots \delta(dx_n^+).$$

The picture number counts the number of δ functions, and one can extend the local results of this paper to the picture number p case, for each $0 < p < n$. This condition on pseudodifferential forms does not, however, yield well-defined geometric objects, as it is coordinate-dependent. One may nevertheless hope that our methods — and the notion of pseudodifferential forms — can be modified to cover the case of arbitrary Lagrangians.

Due to the nature of Berezin integration, the objects of integration on oriented supermanifolds are sections of the Berezinian, $\operatorname{Ber} M$, which transform according to the character $\operatorname{Ber}(T)^{-1}$. The complex of *integral forms* is defined to be the complex of sheaves of $\mathcal{O}_M^\hbar$-modules (we will call this grading the de Rham degree)

$$\Sigma_M^\hbar = \operatorname{Ber}(M)\{0\} \otimes_{\mathcal{O}_M^\hbar} \operatorname{Sym}_{\mathcal{O}_M^\hbar}((T^*M\{-1\})^*),$$

that is, the Berezinian placed in de Rham degree 0 tensored with the commutative algebra generated by the tangent bundle in de Rham degree -1. We have written $(T^*M\{-1\})^*$ instead of $TM\{1\}$ to make it clear that the ghost number of a local generator ∂_{x^i} has ghost number $-\operatorname{gh}(x^i)$ (but the same internal parity). Recall from Remark 9 that the Berezinian is placed in ghost number $2r = 2n + 2\sum_{i=1}^n \operatorname{gh}(x^i)$. We implicitly take the direct sum total complex, summing the ghost number and de Rham degree, as we did with the de Rham complex. Notice that while the de Rham complex is unbounded above on supermanifolds, the complex of integral forms is unbounded below. In particular, there are integral forms of negative de Rham degree.

Explicitly, suppose that we have oriented Darboux coordinates $\{x^i, x_i^+\}$. In these coordinates we may write any (monomial) integral form as

$$\alpha = f(x,x^+)\mathcal{D}(x,x^+) \otimes (\partial_{x^1})^{a_1} \cdots (\partial_{x^n})^{a_n} (\partial_{x_1^+})^{b_1} \cdots (\partial_{x_n^+})^{b_n}, \tag{5.1}$$

where $a_i \in \{0,1\}$ and $b_j \in \mathbb{Z}_{\geqslant 0}$. Notice that, under our grading conventions, this form has de Rham degree $-\sum_{i=1}^n (a_i + b_i)$ and total degree $\operatorname{gh}(f) + 2r - \sum_{i=1}^n (a_i + b_i)$.

Often, especially in the physics literature, one finds notation that involves derivatives of Dirac delta distributions. The same integral form is written in this more intuitive notation as

$$\alpha = \pm f(x,x^+)(dx^1)^{1-a_1} \cdots (dx^n)^{1-a_n} \delta^{(b_1)}(dx_1^+) \cdots \delta^{(b_n)}(dx_n^+).$$

Sometimes $\mathrm{Ber}(M)$ placed in de Rham degree n, to line up with the de Rham degree of ordinary differential forms.

The duality pairing between forms and vector fields provides $\Sigma_M^\hbar$ with the structure of a $\mathrm{dR}_M^\hbar$-module. The action by the generators of $\mathrm{dR}_M^\hbar$ is given, with α as above,

$$\begin{aligned} dx^i\alpha =& (-1)^{|f|+a_1+\cdots+a_{i-1}} a_i f(x,x^+)\mathcal{D}(x,x^+) \\ & \otimes (\partial_{x^1})^{a_1}\cdots\widehat{(\partial_{x^i})^{a_i}}\cdots(\partial_{x^n})^{a_n}(\partial_{x_1^+})^{b_1}\cdots(\partial_{x_n^+})^{b_n} \\ dx_i^+\alpha =& -b_i f(x,x^+)\mathcal{D}(x,x^+) \\ & \otimes (\partial_{x^1})^{a_1}\cdots(\partial_{x^n})^{a_n}(\partial_{x_1^+})^{b_1}\cdots(\partial_{x_i^+})^{b_i-1}\cdots(\partial_{x_n^+})^{b_n} \end{aligned}$$

The sign in the action of dx_i^+ originates from the sign present in the pairing for the double dual of a superspace. From this module structure it is clear how to define the de Rham differential $d = d_\Sigma$ on integral forms:

$$d_\Sigma\left(f(x,x^+)\mathcal{D}(x,x^+)\otimes P(\partial_x,\partial_{x^+})\right) = df\,\mathcal{D}(x,x^+)\otimes P(\partial_x,\partial_{x^+}).$$

Just as in the setting of ordinary differential forms, the odd symplectic form ω provides the complex of integral forms with an extra differential $\Omega = \hbar^{-1}\omega$. The first step in constructing the BV Laplacian on $\Sigma_M^\hbar$ is to compute the cohomology of Ω. The arguments from the differential forms case hold, *mutatis mutandis*. Fix an oriented Darboux chart U. We consider the operator $\Lambda = \hbar\,\iota(\partial_{x^i})\iota(\partial_{x_i^+})$, where the interior multiplication by a vector field is defined to introduce the corresponding vector field on the left in the second tensor factor,

$$\iota(v)\mathcal{D}(x,x^+)\otimes P(\partial_x,\partial_{x^+}) = \mathcal{D}(x,x^+)\otimes vP(\partial_x,\partial_{x^+}).$$

The operator $[\Omega,\Lambda]$ is semisimple on integral forms by the analog of Lemma 12:

$$[\Omega,\Lambda]\alpha = (-n + \deg_{\partial_x}\alpha - \deg_{\partial_{x^+}}\alpha)\alpha.$$

The complex of integral forms splits

$$(\Sigma_M^\hbar(U),\Omega) = \bigoplus_{m=-\infty}^{0}(\Sigma_M^\hbar(U)_m,\Omega)$$

and the cohomology of Ω is concentrated in the rank 1 eigencomplex of eigenvalue $m = 0$,

$$H(\Sigma_M^\hbar(U), \Omega) \cong \mathcal{O}_M^\hbar(U) \cdot \mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}.$$

Suppose now that we have an overlapping oriented Darboux coordinate chart $\{y^i, y_i^+\}$. Then, writing T for the inverse of the change of coordinates matrix with respect to the even-odd polarization and $P = (\partial x/\partial y)$ for the upper-left block, we find that

$$\mathcal{D}(y, y^+) \otimes \partial_{y^1} \cdots \partial_{y^n} + \cdots = \mathcal{D}(x, x^+) \operatorname{Ber}(T) \otimes \partial_{x^1} \cdots \partial_{x^n} \det(P)^{-1} + \cdots$$

where the omitted terms contain $\partial_{x_i^+}$. By Proposition 5 we have $\operatorname{Ber}(T) = (\operatorname{Ber} P)^2$, so the $m = 0$ eigenforms transform as $\operatorname{Ber}(P)$ up to Ω-exact terms. Thus we arrive, via the arguments in the proof of Proposition 8, at the following.

Proposition 30. Let (M, ω) be an odd symplectic supermanifold with $\operatorname{gh}(\omega) = 1$ such that the body $|M|$ of M is oriented. Then, as in Proposition 8, there is a natural isomorphism of sheaves

$$\psi' : H(\Sigma_M^\hbar, \omega) \to |\Lambda_M^\hbar|^{1/2}.$$

This isomorphism sends, locally,

$$f\, |\mathcal{D}(x, x^+)|^{1/2} \mapsto f\, \mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}.$$

Notice that, by Remark 9, ψ' is a map of total degree 0 and even internal parity, as $\mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}$ has de Rham degree $-n$ and ghost number $2r - \sum_{i=1}^n \operatorname{gh}(x^i) = 2n + \sum_{i=1}^n \operatorname{gh}(x^i)$.

The local perturbation theory in the setting of differential forms carries through unchanged. The inclusion i of the $m = 0$ eigenspace into all integral forms fits into a strong deformation retraction as in Proposition 16 where h and p are defined as before. The differential $d = d_\Sigma$ is, for de Rham degree reasons, a small perturbation of the right-hand side $(\Sigma_M^\hbar(U), \Omega)$. The transferred differential on the left-hand side, $(\Sigma^\hbar_M(U)_{m=0}, 0)$ is computed, if $\alpha = f\,\mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}$, as

$$
\begin{aligned}
pdhdi\alpha &= pd\Lambda\left(dx^k \frac{\partial f}{\partial x^k}\mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n}\right) \\
&= -\hbar pd\left(\frac{\partial f}{\partial x^k}\mathcal{D}(x, x^+) \otimes \partial_{x^+_k}\partial_{x^1} \cdots \partial_{x^n}\right) \\
&= \hbar\,\frac{\partial^2 f}{\partial x^+_k \partial x^k}\mathcal{D}(x, x^+) \otimes \partial_{x^1} \cdots \partial_{x^n},
\end{aligned}
\tag{5.2}
$$

as desired. These computations show that, on U, half-densities with the BV Laplacian can be placed into a strong deformation retraction with integral forms on right-hand side.

$$
(|\Lambda^\hbar_M|^{1/2}(U), \hbar\,\Delta) \underset{p'}{\overset{i'}{\rightleftarrows}} (\Sigma^\hbar_M(U), \Omega + d_\Sigma) \circlearrowleft h'
$$

The globalization arguments of Chapter 4 carry over as well. The map i is not a map of complexes, so we transfer the Čech differential on the (direct product total) Čech complex of integral forms to that of half-densities (see Proposition 27). Globalization is now achieved by the analog of Proposition 29: we perturb the right by $d = d_\Sigma$ and obtain

$$
(\mathrm{Tot}^*\,\check{C}(\mathcal{U}, |\Lambda^\hbar_M|^{1/2}), \check{d} + p'A'i') \underset{p''}{\overset{i''}{\rightleftarrows}} (\mathrm{Tot}^*\,\check{C}(\mathcal{U}, \Sigma^\hbar_M), (-1)^p(\Omega + d_\Sigma) + \check{d}) \circlearrowleft h''
$$

where $p'A'i'$ is computed locally to be nothing more than $\hbar\,\Delta$. As the Čech differential commutes with the BV Laplacian, we obtain the analog of Theorem 24.

Theorem 31. The local expression $\hbar\,\partial^2/\partial x^i \partial x^+_i$ for the BV Laplacian acting on integral forms in Equation 5.2 globalizes to a differential on the sheaf of half-densities $|\Lambda^\hbar_M|^{1/2}$.

The map i', in particular, gives us a local identification of half-densities with integral forms, intertwining the BV Laplacian $\hbar\,\Delta$ with $\Omega + d_\Sigma$.

Bibliography

[1] I. A. Batalin and G. A. Vilkovisky. "Gauge algebra and quantization". In: *Phys. Lett. B* 102.1 (1981), pp. 27–31.

[2] Alexander Belopolsky. "New geometrical approach to superstrings". In: (Mar. 1997). arXiv: hep-th/9703183.

[3] I. N. Bernštein and D. A. Leites. "How to integrate differential forms on supermanifolds". In: *Funkcional. Anal. i Priložen.* 11.3 (1977), pp. 70–71.

[4] I. N. Bernštein and D. A. Leites. "Integral forms and the Stokes formula on supermanifolds". In: *Funkcional. Anal. i Priložen.* 11.1 (1977), pp. 55–56.

[5] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331.

[6] R. Brown. "The twisted Eilenberg-Zilber theorem". In: *Simposio di Topologia (Messina, 1964)*. Edizioni Oderisi, Gubbio, 1965, pp. 33–37.

[7] R. Catenacci et al. "On forms, cohomology and BV Laplacians in odd symplectic geometry". In: *Lett. Math. Phys.* 111.2 (2021), Paper No. 44, 32.

[8] M. Crainic. "On the perturbation lemma, and deformations". In: *arXiv Mathematics e-prints*, math/0403266 (Mar. 2004), math/0403266. arXiv: math/0403266 [math.AT].

[9] Ezra Getzler and Sean Weinz Pohorence. *Global gauge conditions in the Batalin-Vilkovisky formalism*. 2020. arXiv: 1911.11269 [math-ph].

[10] V. K. A. M. Gugenheim. "On the chain-complex of a fibration". In: *Illinois J. Math.* 16 (1972), pp. 398–414.

[11] Hovhannes M. Khudaverdian. "Semidensities on odd symplectic supermanifolds". In: *Comm. Math. Phys.* 247.2 (2004), pp. 353–390.

[12] Hovhannes M. Khudaverdian and Theodore Th. Voronov. “Differential forms and odd symplectic geometry”. In: *Geometry, topology, and mathematical physics*. Vol. 224. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 2008, pp. 159–171.

[13] D. A. Leites. “Introduction to the theory of supermanifolds”. In: *Uspekhi Mat. Nauk* 35.1(211) (1980), pp. 3–57, 255.

[14] Yuri I. Manin. *Gauge field theory and complex geometry*. Second. Vol. 289. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the 1984 Russian original by N. Koblitz and J. R. King, With an appendix by Sergei Merkulov. Springer-Verlag, Berlin, 1997, pp. xii+346.

[15] Tony Pantev et al. “Shifted symplectic structures”. In: *Publ. Math. Inst. Hautes Études Sci.* 117 (2013), pp. 271–328.

[16] Albert Schwarz. “Geometry of Batalin-Vilkovisky quantization”. In: *Comm. Math. Phys.* 155.2 (1993), pp. 249–260.

[17] Pavol Ševera. “On the origin of the BV operator on odd symplectic supermanifolds”. In: *Lett. Math. Phys.* 78.1 (2006), pp. 55–59.

[18] Weishu Shih. “Homologie des espaces fibrés”. In: *Inst. Hautes Études Sci. Publ. Math.* 13 (1962), p. 88.

[19] Theodore Voronov. “Graded manifolds and Drinfeld doubles for Lie bialgebroids”. In: *Quantization, Poisson brackets and beyond (Manchester, 2001)*. Vol. 315. Contemp. Math. Amer. Math. Soc., Providence, RI, 2002, pp. 131–168.