THE ATIYAH-SINGER INDEX THEOREM

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The main reference for this talk is [BGV04].

The Atiyah-Singer index theorem is fundamentally a theorem about Dirac operators. These are differential operators named after Dirac, who originally was looking for an operator that was the square root of the Laplacian (in Minkowski spacetime), in order to write down an evolution equation for the relativistic quantum mechanical particle.

Definition 1. Let $E \to M$ be a $\mathbb{Z}/2$-graded vector bundle $E^+ \oplus E^- \to M$ on a Riemannian manifold M. A **Dirac operator** is a first order differential operator of odd parity

$$\mathsf{D}^\pm : \Gamma(M, E^\pm) \to \Gamma(M, E^\mp)$$

that squares to a generalized Laplacian (an even second-order differential operator whose principal symbol is that of the Laplacian). We will generally assume that our Dirac operators are selfadjoint.

Dirac operators are ubiquituous, but I will focus on one specific example that is familiar from geometry.

Example 2. Let M be a Riemannian manifold and let E be the exterior algebra on the cotangent space, $E = \Lambda^\bullet T^*M$. This bundle is naturally $\mathbb{Z}$-graded, but we will give it instead the induced $\mathbb{Z}/2$-grading, where E^+ consists of the even wedges and E^- consists of the odd wedges. In this case there is a natural odd-parity differential operator,

$$\mathsf{D} = d + d^*,$$

as d raises the $\mathbb{Z}$-degree by 1 and d^* decreases it by one. This operator of course squares to the Laplacian, so D is a Dirac operator.

The Atiyah-Singer index theorem is, roughly, an equality between an analytic quantity associated to a Dirac operator, and a cohomological quantity associated to $E \to M$.

Theorem 3 (Atiyah-Singer)**.** *Let M be a compact oriented Riemannian manifold of even dimension n. Then*

$$\operatorname{ind} \mathsf{D} = (2\pi i)^{-n/2} \int_M \hat{A}(M) \operatorname{ch}(E/S).$$

Let's look at what these objects are. The left-hand side is easy to define. The index of a Dirac operator is defined to be

$$\operatorname{ind} \mathsf{D} = \dim \ker \mathsf{D}^+ - \dim \ker \mathsf{D}^-.$$

It is a good exercise to check that this lines up with the usual (ungraded) definition of index. Heuristically, one should think that the dimension of the solution space of a differential equation can be difficult to compute, but the index, according to the theorem is tightly constrained by the topology of the setup.

The right hand side is the integral of a certain cohomology class on M. Since we will not explicitly be computing these quantities, I won't define them completely. Let me just give a formula for $\hat{A}(M)$. Write R for the curvature of the Levi-Civita connection; then

$$\hat{A} = \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2} \in \Omega^{4\bullet}(M).$$

This is a (natural) inhomogeneous differential form on our manifold.

Let's return to our example of $\mathsf{D} = d + d^*$ on the exterior algebra of T^*M.

Example 4. Let's compute the index of $\mathsf{D} = d + d^*$. Since D is selfadjoint, we have that $\ker \mathsf{D} = \ker \mathsf{D}^2$. The index is

$$\operatorname{ind} \mathsf{D} = \dim \ker \mathsf{D}^+ - \dim \ker \mathsf{D}^-$$

but the first term is the dimension of the even-degree harmonic forms and the second term is the dimension of the odd-degree harmonic forms. But recall now that the Hodge theorem tells us that the harmonic forms represent cohomology, whence we conclude that

$$\operatorname{ind} \mathsf{D} = \chi(M).$$

Computing the right-hand side of the Atiyah-Singer index theorem is more involved. If you do a nasty compute in local coordinates, it turns out that

$$\hat{A}(M)\operatorname{ch}(E/S) = \operatorname{Pf}(-R)$$

where R is the curvature of the Levi-Civita connection. Recall that Pf is the Pfaffian of an antisymmetric even-dimensional matrix, which is a square root of the determinant. The Atiyah-Singer index theorem now reduces to the well-known Gauss-Bonnet-Chern theorem:

$$\chi(M) = (2\pi)^{-n/2}\int_M \operatorname{Pf}(-R).$$

Among other well-known theorems that one can derive from the Atiyah-Singer index theorem are the Hirzebruch signature theorem, the index theorem for the Dirac operator on a (twisted) spinor bundle on a spin manifold, and the Riemann-Roch-Hirzebruch theorem.

There have been many many proofs of the Atiyah-Singer index theorem over the years. The one I am familiar with is fairly elementary (requires no K-theory, etc.) and the main tool is the heat kernel for generalized Laplacians D^2. The proof proceeds roughly in two steps. The first is the McKean-Singer formula, which writes

$$\operatorname{ind} \mathsf{D} = \operatorname{str} e^{-t\mathsf{D}^2}.$$

Here we are using the supertrace which is the difference of the trace on the even and odd spaces. This is actually very easy to prove once you know the spectrum of D^2 (one checks that the dimensions of the even and odd nonzero-eigenspaces cancel as D is an isomorphism there, leaving only the kernels). Notice that the left-hand side is independent of t. The second part of the proof precisely leverages this t-independence and computes

$$\lim_{t\to 0} \operatorname{str} e^{-t\mathsf{D}^2} = \lim_{t\to 0}\int_M \operatorname{str} p_t(x,x)dx$$

using the small-time asymptotics of the heat kernel as well as some clever Clifford algebra tricks. This computation is done in normal coordinates and is miraculously found to recover the topological quantities on the right-hand side. This is known as the "local" Atiyah-Singer index theorem (due to Ezra).

References

[BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.