SOME BACKGROUND ON GEOMETRIC QUANTIZATION

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1. Introduction

Let M be a compact oriented 3-manifold. Chern-Simons theory with gauge group G (that we will take to be compact, connected, and simply-connected) on M is the data of a principal G-bundle $\pi : P \to M$ together with a *Lagrangian density* $\mathscr{L} : \mathscr{A} \to \Omega^3(P)$ on the space of connections $\mathscr{A}$ on P given by

$$\mathscr{L}_{\mathrm{CS}}(A) = \langle A \wedge F\rangle + \frac{2}{3}\langle A \wedge [A \wedge A]\rangle.$$

Let us detail the notation used here. Recall first that a connection $A \in \mathscr{A}$ is a G-invariant $\mathfrak{g}$-valued one-form, i.e. $A \in \Omega^1(P;\mathfrak{g})$ such that $R_g^* A = \mathrm{Ad}_{g^{-1}} A$, satisfying the additional condition that if $\xi \in \mathfrak{g}$ then $A(\xi_P) = \xi$ if ξ_P is the vector field associated to ξ. Notice that $\mathscr{A}$, though not a vector space, is an affine space modelled on $\Omega^1(M, P \times_G \mathfrak{g})$. The curvature F of a connection A is is the $\mathfrak{g}$-valued two-form given by $F(v, w) = dA(v_h, w_h)$, where $\bullet_h$ denotes projection onto the horizontal distribution $\ker \pi_*$.[1] Finally, by $\langle -, -\rangle$ we denote an ad-invariant inner product on $\mathfrak{g}$.

The Chern-Simons action is now given

$$S_{\mathrm{SC}}(A) = \int_M \mathscr{L}_{\mathrm{SC}}(A)$$

and the quantities of interest are expectation values of observables $\mathscr{O} : \mathscr{A} \to \mathbb{R}$

$$\langle \mathscr{O}\rangle = \int_{\mathscr{A}/\mathscr{G}} \mathscr{O}(A) e^{iS_{\mathrm{SC}}(A)/\hbar}.$$

Here $\mathscr{G}$ is the group of automorphisms of $P \to \Sigma$, which acts by pullback on $\mathscr{A}$ − the physical states are unaffected by these gauge transformations, so we integrate over the quotient $\mathscr{A}/\mathscr{G}$ to eliminate the redundancy. In [Wit89] Witten consider the

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[1]Here $[- \wedge -]$ and d are given by a combination of the wedge product and commutator.

following observable: let C be a closed oriented curve in M and V be an irreducible representation of G. Then we define the *Wilson line*

$$W_{C,V}(A) = \mathrm{tr}_V \exp \int_C A.$$

Witten computed the expectation values

$$\langle \prod_i W_{C_i,V_i} \rangle$$

for (C_i, V_i) pairs of curves and G-irreps, and recovered in the case of $M = S^3$ the Jones polynomial and its generalizations. Moreover, taking M to be an arbitrary 3-manifold and taking no curves, we obtain invariants of 3-manifolds that are effectively computable.

The goal of these notes is to provide some background on geometric quantization which Pyongwon will use to tell us about a rigorous procedure for obtaining quantum Chern-Simons theory on a compact oriented 2-manifold Σ without resorting to the path integral formalism. To do this, we will use *geometric quantization*, a method of quantizing a symplectic manifold to obtain a Hilbert space. We will follow the constructions of [ADPW91].

2. Classical versus quantum

Recall that the data of a classical mechanical system can be encoded as symplectic geometry. A symplectic form on a manifold M is a nondegenerate closed two-form $\omega \in \Omega^2(M)$. By nondegenerate we simply mean that ω_p is a nondegenerate skew-symmetric bilinear form for each $p \in M$, or more globally, that ω induces an isomorphism $TM \to T^*M$. A symplectic manifold is then a pair (M, ω), and it is not hard to see that $\dim M$ must be even. Let us fix some notation: by X_f we mean the unique vector field corresponding to the one-form df:

$$\iota_{X_f}\omega = df.$$

Then we have a Poisson bracket $\{-,-\}$ on $C^\infty(M)$ given by

$$\{f, g\} = \omega(X_f, X_g),$$

under which $C^\infty(M)$ forms a Lie algebra. Notice that the bracket is also a biderivation. We say that $C^\infty(M)$ forms a Poisson algebra (over $\mathbb{R}$).

Consider, for concreteness, a free particle in $\mathbb{R}^n$. The associated symplectic manifold $(\mathbb{R}^{2n}, \sum dq^i \wedge dp^i)$ represents the phase space of the system – all posible states (q, p) of the particle. The observables in this formulation are simply smooth functions on M. The energy, for instance, is given $H(q, p) = |p|^2/2m$.

In quantum mechanics, on the other hand, the phase space is given as a (complex) Hilbert space $\mathcal{H}$ (or more precisely the projectivized space $\mathbb{P}\mathcal{H}$) and observables correspond to selfadjoint operators. In particular, one computes the expectation value of a given observable as

$$\langle \mathcal{O} \rangle = \int_{\mathcal{H}} \mathcal{O}(\psi) e^{iS(\psi)/\hbar},$$

where $S : \mathcal{H} \to \mathbb{R}$ is the action of the system.

Notice that there is a canonical procedure for obtaining a classical system from a quantum one: take $\hbar \to 0$. Very roughly speaking, as $\hbar$ becomes small the exponential in the integral above oscillates wildly and the integral is dominated by contributions from the classical locus $\delta S = 0$.

The problem of quantization, then, is the converse question: does a classical system determine a quantum system? This is an interesting question to ask because often in physics one starts with a classical theory such as electromagnetism (a classical field theory) and wishes to obtain a quantum theory such as quantum electrodynamics (a quantum field theory). Unfortunately, as the saying goes, "quantization is an art, not a functor." But let us be more precise and describe exactly what we mean by quantization (at least for our purposes).

Definition 1 (Dirac)**.** Let (M, ω) be a symplectic manifold. A *quantization* of M is a complex Hilbert space $(\mathcal{H}, \mathcal{O})$ with selfadjoint operators $\mathcal{O}$, together with an $\mathbb{R}$-linear map $\hat{\bullet} : C^\infty(M) \to \mathcal{O}$ such that $\hat{1}$ is the identity operator on $\mathcal{H}$ and

$$[\hat{f}, \hat{g}] = -i\hbar \widehat{\{f, g\}}.$$

Unfortunately, it is unclear how to do this in general without making $\mathcal{H}$ unphysically large.[2]

Before we begin discussing the procedure of geometric quantization, which will approximate the notion of quantization above, let us see how it applies to the case of Chern-Simons theory. Recall that the phase space of Chern-Simons theory is the space $\mathscr{A}^\flat/\mathscr{G}$ of flat connections on Σ up to gauge transformation. There is a natural symplectic structure on $\mathscr{A}^\flat$ inherited from the symplectic structure of $\mathscr{A}$ via Marsden-Weinstein reduction. Since $\mathscr{A}$ is an affine space modelled on the vector space $\Gamma(\Sigma, P \times_G \mathfrak{g})$, the tangent space to $\mathscr{A}$ at any connection is said vector space. There is thus a natural symplectic form on $\mathscr{A}$ given

$$\omega_{\mathscr{A}}(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle.$$

In order to describe how this symplectic form descends to $\mathscr{A}^\flat$, we recall some details of symplectic reduction. Let G be a Hamiltonian group action on (M, ω). That is, the G action satisfies:

(1) G acts through symplectomorphisms;
(2) if $\xi \in \mathfrak{g}$, the one form associated to the vector field ξ_M is exact:
$$\iota_{\xi_M}\omega = d\kappa(\xi),$$
for $\kappa(\xi) \in C^\infty(M)$;
(3) the associated comoment map $\kappa : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism.

Then there exists a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$ determined by

$$\kappa(\xi)(p) = \mu(p)(\xi).$$

The fundamental result of symplectic reduction is that if a Hamiltonian action of G is free and proper then $\mu^{-1}(0)/G_0$ is a symplectic manifold with symplectic form ω_0 uniquely characterized by $\pi_0^*\omega_0 = \iota_0^*\omega$.

Let us return to Chern-Simons theory.

Exercise 2. Make sense of the statement that $F : \mathscr{A} \to \Omega^2(\Sigma, P \times_G \mathfrak{g})$ provides a moment map for the $\mathscr{G}$-action on $\mathscr{A}$.

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[2] In fact, there are various no-go theorems in the literature, c.f. Gronewald-van Hove.

In view of the exercise above the theory of reduction yields a symplectic structure on the moduli of flat connections $F^{-1}(0)/\mathscr{G} = \mathscr{A}^\flat/\mathscr{G}$. Thus the story of geometric quantization is indeed applicable.[3]

3. Geometric quantization

In this section we follow [Woo92] and [Hal13].

The first step in the geometric quantization of a symplectic manifold (M, ω) is *prequantization*, which assigns to M a line bundle with connection whose curvature is ω. The prequantum Hilbert space is then taken to be the square-integrable sections of this line bundle.

There is, of course, an obvious line bundle: the trivial one. Can we get away with this? Consider the space $L^2(M)$ of square-integrable smooth complex functions on M. This space has a natural inner product given by

$$\langle \psi, \psi' \rangle = \int_M \bar{\psi}\psi' \varepsilon,$$

where $\varepsilon = (\omega/2\pi\hbar)^n$ is a volume form. There is an obvious quantization of $f \in C^\infty(M)$ to

$$\psi \mapsto -i\hbar X_f \psi.$$

Unfortunately this will send constants to the zero operator. There is an immediate correction:

$$\psi \mapsto (-i\hbar X_f + f)\psi.$$

This quantization is no longer a Lie algebra homomorphism, so we add yet another term

$$\psi \mapsto (-i\hbar X_f + f - i\iota_{X_f}\lambda/\hbar)\psi,$$

where λ is a one-form such that $d\lambda = \omega$. This prescription works, but now depends on λ, which need not exist in general. The way out is to replace the trivial bundle with a Hermitian bundle together with a connection whose curvature is ω. The key fact is as follows.

Theorem 3. *There exists a Hermitian line bundle $L \to M$ and a Hermitian connection $\nabla : \mathrm{H}^0(M, L) \to \mathrm{H}^0(M, L \otimes T^*M)$ on L with curvature $\hbar^{-1}\omega$ if and only if $(2\pi\hbar)^{-1}\omega \in \mathrm{H}^2(M, \mathbb{Z})$. In this case, the choice of (L, ∇) is parameterized by $\mathrm{H}^1(M, U(1))$.*

Using this, we can define prequantization.

Definition 4. Let (M, ω) be a integral symplectic manifold. Then a *prequantum bundle* is a choice of line bundle $L \to M$ and connection ∇ on L with curvature $\hbar^{-1}\omega$. The *prequantum Hilbert space* is the space $\mathcal{H}^{\mathrm{pre}}$ of square-integrable sections of L together with the obvious inner product. The operator associated to $f \in C^\infty(M)$ is given by

$$\hat{f}\psi = (-i\hbar\nabla_{X_f} + f)\psi.$$

Example 5. Consider $M = \mathbb{R}^{2n}$ with coordinates (q, p) and its usual symplectic form $\omega = dq^i \wedge dp^i$. The integrality of ω is clear because M is exact, $\omega = d(p_i dq^i) = d\lambda$. In this case the trivial line bundle with connection $\nabla_v = v + \iota_v\lambda$ provides a

[3] Exercise: why is the form integral?

prequantum line bundle. Notice that $X_{q^i} = \partial/\partial p^i$ and $X_{p_i} = -\partial/\partial q^i$. Hence we find

$$\hat{q}^i \psi = \left(q^i - i\hbar \frac{\partial}{\partial p_i} \right) \psi$$
$$\hat{p}_i \psi = i\hbar \frac{\partial}{\partial q^i} \psi.$$

A straightforward computation reveals that $[\hat{q}^i, \hat{p}_i] = -i\hbar = -i\hbar \widehat{\{q^i, p_i\}}$, as desired. On the other hand, there is something strange going on: our "wavefunctions" depend on both p and q, which is why $\hat{q}^i$ has an unfamiliar $\partial/\partial p_i$ term. Usually in quantum mechanics, we work with wavefunctions depending only on the q^i or only on the p_i, with the two perspectives related via the Fourier transform.

The previous example shows that, even in the case of a particle in $\mathbb{R}^{2n}$, the prequantum Hilbert space constructed by prequantization is morally twice as large as it should be. The next step of geometric quantization, polarization, restricts the space of functions on M that we quantize. On T^*M, for example, one has the vertical and horizontal polarizations, which yield the usual position and momentum Hilbert spaces. Polarization is a rather delicate and involved procedure, so we only sketch the beginning of the story.

Definition 6. A *polarization* on (M, ω) is an integrable Lagrangian subbundle $\mathscr{P} \hookrightarrow TM \otimes \mathbb{C}$. Then, if $L \to M$ is a prequantum line bundle and $\mathscr{P}$ is a polarization, we define the *quantum Hilbert space* $\mathcal{H}$ to be the set of square integrable sections that are covariantly constant along $\mathscr{P}$, i.e.

$$\mathcal{H} = \{s \in L^2(M, L) \mid \nabla_X s = 0, X \in \mathscr{P}\}.$$

Example 7. Consider $M = T^*\mathbb{R} = \mathbb{R}^2$ with coordinates (q, p) and the trivial prequantum line bundle $(L = M \times \mathbb{C}, \nabla_v = v + i\iota_v \lambda)$, where $\lambda = pdq$. There is an obvious polarization of $\mathbb{R}^2$ by contangent fibers so that the polarized sections satisfy $\nabla_{\partial/\partial p} \psi = 0$, i.e.

$$\left(\frac{\partial}{\partial p} + i\iota_{\partial/\partial p} pdq \right) \psi = \frac{\partial \psi}{\partial p} = 0.$$

Hence $\mathcal{H}$ consists of square-integrable sections depending on q. Unfortunately there are no such sections as

$$\int_{\mathbb{R}^2} \overline{\psi(x)} \psi(x) \; \mathrm{d}x \, \mathrm{d}p = \infty$$

unless $\psi = 0$.

Alternatively one could choose another polarization, taking the zero section and its translates, upon which the polarized sections satisfy $\nabla_{\partial/\partial q} \psi = 0$, which one can solve to find that $\psi(q, p) = \psi(p, 0) e^{-iqp}$. Again, these will not be square-integrable.

To actually obtain square-integrable sections we would, roughly speaking, have to cut down our base manifold from $T^*\mathbb{R}$ to $\mathbb{R}$. Hence even after polarization we do not necessarily obtain the Hilbert spaces familiar to physicists.

Remark 8. Polarizations need not exist! There is a class of compact 4-dimensional symplectic manifolds admitting no polarizations at all, c.f. [Got87].

Recall that a Kähler manifold (M, J, ω) is a symplectic manifold together with an integrable almost complex structure $J : TM \to TM$ (with $J^2 = -\operatorname{id}$) compatible with the symplectic structure: $\omega(Jv, Jw) = \omega(v, w)$. The complexified tangent bundle now splits $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ along the eigenvalues $\pm i$ of $J \otimes \mathbb{C}$. The canonical *Kähler polarization* on (M, J, ω) is simply the subbundle $T^{1,0}M \subset TM \otimes \mathbb{C}$, i.e. the holomorphic tangent bundle.

Now, prequantization of (M, J, ω) yields a hermitian line bundle $L \to M$ equipped with a hermitian connection ∇ with curvature given by the Kähler form ω. The connection extends linearly to $\nabla^{\mathbb{C}} : \mathrm{H}^0(M, L) \to \mathrm{H}^0(M, L \otimes T^*M \otimes \mathbb{C})$, and we can define the *Kähler polarized sections* of L to be $s \in \mathrm{H}^0(M, L)$ such that $\nabla_X s = 0$ for all $X \in \mathrm{H}^0(M, T^{0,1}M)$, that is, sections that are covariantly holomorphic. The *quantum Hilbert space* $\mathcal{H}$ is then the space of square-integrable Kähler polarized sections.

Exercise 9. Show that, in the Kähler case, L has a natural holomorphic structure, and $\mathcal{H}$ is simply the space of square-integrable holomorphic sections.

There are a number of undesirable features of this story. While it can be shown that there always exist nonzero local polarized sections, there need not always exist global square-integrable polarized sections. Moreover, the question of what the operators on $\mathcal{H}$ are now becomes quite subtle. For instance, it only makes sense to consider the subset of the prequantization operators preserving $\mathcal{H} \subset \mathcal{H}^{\mathrm{pre}}$.

Example 10. Consider $M = \mathbb{R}^2$ with complex coordinates $z = q + ip, \bar{z} = q - ip$. Recall that the prequantum line bundle in this case is the trivial line bundle with connection $\nabla_v = v + \iota_v \lambda$, where $\lambda = pdq$. Some computation reveals that the polarized states are of the form

$$\psi(q, p) = F(z)e^{-p^2/2\hbar} = F(z)e^{-\mathrm{Im}(z)^2/\hbar},$$

where F is an arbitrary holomorphic function. Those ψ which are square-integrable form what is known to functional analysts as the Segal-Bargmann Hilbert space.

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