Differential forms on stacks

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The references for these lectures are available here.

Introduction

This series of lectures discusses differential forms on stacks, by which we mean simply simplicial manifolds modulo a certain equivalence relation. (Later, we will impose some natural geometric conditions on these simplicial manifolds.)

Why **simplicial manifolds**? The simple answer is that they yield a general framework for differential geometry. For example, they include (in a slightly unfamiliar language) **Lie groupoids**. In this way, they provide a language for discussing differentiable dynamical systems, foliations, and characteristic classes. This story was developed in the years around 1970 by Bott, Gelfand, Fuks, Haefliger and Segal, among others. We will be particularly interested in the approach to Chern-Weil theory contained in Shulman's 1972 Berkeley thesis.

But it turns out that this is just the tip of the iceberg. The de Rham complex, and Chern-Weil theory, may be extended to a far more general class of simplicial manifolds: the **higher Lie groupoids**.

Plan of lectures:

1. Simplicial manifolds and nerves of Lie groupoids
2. Differential forms on simplicial manifolds — Shulman's construction of closed differential forms on the nerve of a Lie group
3. cohomological descent (quasi-invariance under Morita equivalence)
4. symplectic forms on higher Lie groupoids

If time had permitted, we would have gone on to discuss two important extensions:

5. derived stacks (studied by Pantev, Toën, Vaquié and Vezzosi), in which simplicial manifolds are replaced by simplicial differential graded manifolds;
6. the Cheeger-Simons theory of differential characters, which generalizes the Kostant-Souriau theorem that every closed two-form with periods in $2\pi i\mathbb{Z}$ is the curvature of a Hermitian line bundle to higher degrees.

Groupoids

Definition

A **groupoid** $\mathcal{G}$ is a category in which every morphism has a (unique) inverse.

It is possible, but not very enlightening, to write out the definition in full. The nice thing about the formalism of simplicial sets (manifolds) is that it produces a far more conceptual axiomatization of (Lie) groupoids.

Axioms for groupoids

Data

- Sets $\mathcal{G}_0$ (objects) and $\mathcal{G}_1$ (morphisms)
- Functions $s : \mathcal{G}_1 \to \mathcal{G}_0$ (source) and $t : \mathcal{G}_1 \to \mathcal{G}_0$ (target)
- A function $m : \mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ (product)
- A function $e : \mathcal{G}_0 \to \mathcal{G}_1$ (identity)
- A function $i : \mathcal{G}_1 \to \mathcal{G}_1$ (inverse)

Relations

Let $x, y, \ldots \in \mathcal{G}_0$ and $a, b, \ldots \in \mathcal{G}_1$.

- $s(e(x)) = t(e(x)) = x$ and $s(i(a)) = t(a)$
- $s(m(a,b)) = s(b)$ and $t(m(a,b)) = t(a)$
- $m(a, m(b,c)) = m(m(a,b),c)$ (associativity)
- $m(a, e(s(a))) = m(e(t(a)), a) = a$ (identity)
- $i(e(x)) = e(x)$, $i(i(a)) = a$, and $m(i(a), a) = m(a, i(a)) = e$ (inverse)

Surjective submersions

Definition

A **submersion** $f : M \to N$ is a differentiable map such that

$$T_p f : T_p M \to T_{f(p)} N$$

is surjective for each $p \in M$. A **surjective submersion** is a submersion which is surjective as a function of sets.

We will use the following properties of surjective submersions:

1. identity maps are surjective submersions;
2. surjective submersions are closed under composition;
3. pullbacks of surjective submersions exist, and are again surjective submersions;
4. if gf and f are surjective submersions, then g is a surjective submersion.

Grothendieck topologies

Speaking somewhat loosely, we will call a subcategory $\mathcal{P} \subset \mathcal{C}$ of a category $\mathcal{C}$ satisfying these four axioms a (Grothendieck) **topology** on $\mathcal{C}$. In the general setting, morphisms of $\mathcal{P}$ will be referred to as **covers**. (Axiom 4 is not usually assumed, but it happens to be true in all cases of geometric interest, and is important for the development of the abstract theory of Lie groupoids.)

Other examples of topologies:

- surjective étale maps ($T_p f$ is an isomorphism for each $p \in M$) — this topology is used in the theory of orbifolds
- surjective proper maps — this topology is used by Deligne in the construction of a mixed Hodge structure on the cohomology of a quasi-projective algebraic variety over $\mathbb{C}$.

Lie groupoids

Lie groupoids (*groupoïdes différentiables*) were introduced by Charles Ehresmann in a talk in Brussels in 1958.

Definition

A **Lie groupoid** is a groupoid $\mathcal{G}$ in which

- the spaces of objects $\mathcal{G}_0$ and morphisms $\mathcal{G}_1$ are manifolds;
- all of the structure maps are differentiable;
- the source and target maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are **surjective submersions**.

We may also consider variants of Ehresmann's axioms, in which manifolds are replaced by analytic varieties, Banach analytic varieties or algebraic schemes, or more generally, by any category $\mathcal{C}$ with a subcategory of covers $\mathcal{P}$ defining a topology.

Examples of Lie groupoids: the Čech groupoid

If $U \to X$ is a surjective submersion, then

$$\mathcal{G}_0 = U, \qquad \mathcal{G}_1 = U \times_X U,$$

is a Lie groupoid (the Čech groupoid of the submersion). The operations defining the groupoid are as follows:

$$s(x,y) = y, \quad t(x,y) = x, \quad e(x) = (x,x), \quad m((x,y),(y,z)) = (x,z).$$

In particular, if $U = \coprod_{i \in I} U_i$, where $\{U_i\}_{i \in I}$ is an open cover of X, we get the Čech groupoid in the traditional sense:

$$\mathcal{G}_0 = \coprod_{i \in I} U_i, \qquad \mathcal{G}_1 = \coprod_{i,j \in I} U_i \cap U_j.$$

Examples of Lie groupoids: action groupoids

If G is a Lie group acting differentiably on a manifold M, then $\mathcal{G} = [M/G]$ is the Lie groupoid with

$$\mathcal{G}_0 = M, \qquad\qquad \mathcal{G}_1 = M \times G,$$

and operations

$$s(x,g) = x, \quad t(x,g) = gx, \quad e(x) = (x,1), \quad m((hx,g),(x,h)) = (x,gh).$$

Exercise

If the action of G on M is locally trivial, with Hausdorff quotient $X = M/G$, then the action groupoid $[M/G]$ is isomorphic to the Čech nerve of the surjective submersion $M \to X$.

Simplices

The abstract n-simplex $[n]$ $(n \geqslant 0)$ is the finite category

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n$$

with objects $\{i \mid 0 \leqslant i \leqslant n\}$ and morphisms $\{i \to j \mid 0 \leqslant i \leqslant j \leqslant n\}$. A functor from $[m]$ to $[n]$ is prescribed by a (weakly) monotone function on the totally ordered set of objects.

Definition

The category Δ is the category whose objects are the categories $\{[n] \mid n \geqslant 0\}$ and whose morphisms are the functors between them.

One might have thought that Δ should be a 2-category. In fact, there are no non-trivial natural transformations between two functors $[m] \to [n]$.

Simplicial objects

Definition

A **simplicial object** $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ in a category $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, that is, a presheaf on Δ with values in $\mathcal{C}$.
A **simplicial morphism** is a natural transformation of functors in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$.

It is customary to write $s\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$. For example, if $\mathcal{S}$ and $\mathcal{M}$ are the categories of sets and differentiable manifolds respectively, then $s\mathcal{S}$ and $s\mathcal{M}$ are the categories of simplicial sets and manifolds.

The value of a simplicial object $X_\bullet$ at $[n]$ is denoted X_n.

The nerve of a groupoid

Definition

The **nerve** $N_\bullet\mathcal{G} \in s\mathcal{S}$ of a groupoid $\mathcal{G}$ is the simplicial set

$$N_n\mathcal{G} = \mathrm{Fun}([n], \mathcal{G}).$$

In other words, an n-simplex of the nerve is a sequence of objects

$$(x_0, \dots, x_n) \in \mathcal{G}_0^{n+1}$$

and morphisms $g_{ij} : x_i \to x_j$, $0 \leqslant i \leqslant j \leqslant n$. (It suffices to specify the morphisms $(g_1, \dots, g_n)$, where $g_i = g_{i-1,i}$.) For example, if G is a group, then

$$N_n G \cong G^n.$$

We see that $N_0\mathcal{G} = \mathcal{G}_0$, $N_1\mathcal{G} = \mathcal{G}_1$, and

$$N_{n+1}\mathcal{G} = N_n\mathcal{G} \times_{\mathcal{G}_0} \mathcal{G}_1.$$

Reconstructing a groupoid from its nerve

The simplicial set Δ^k is defined by

$$\Delta^k_n = \Delta([n],[k]).$$

The n-simplices of Δ^k are sequences $0 \leqslant i_0 \leqslant \cdots \leqslant i_n \leqslant k$; denote this simplex by $[i_0 \dots i_n]$. It is **nondegenerate** if $0 \leqslant i_0 < \cdots < i_n \leqslant k$.

A finite simplicial complex $K_\bullet$ is a simplicial set which may be realized as a simplicial subset of Δ^k for some k.

Examples

- The ith face $\partial_i\Delta^k \subset \Delta^k$ is the simplicial set of all simplices not containing the ith vertex.
- The **boundary** $\partial\Delta^k \subset \Delta^k$ is the union of all of the faces $\partial_j\Delta^k$, $j \in \{0,\dots,k\}$.
- The **horn** $\Lambda^k_i \subset \Delta^k$ is the union of the faces $\partial_j\Delta^k$, $j \neq i$.

Matching spaces

If $X_\bullet$ is a simplicial set, and $K_\bullet$ is a finite simplicial complex, we define $\operatorname{Hom}(K, X)$ to be the set of all simplicial maps between $K_\bullet$ and $X_\bullet$. More explicitly, $\operatorname{Hom}(K, X)$ is the finite limit (fibred product)

$$\operatorname{Hom}(K, X) = \text{equalizer}\left(\prod_{k=0}^{\infty} \prod_{\substack{\text{nondegenerate} \\ [i_0 \ldots i_k] \in K_k}} X_k \right.$$

$$\left. \rightrightarrows \prod_{\substack{\text{monomorphisms} \\ [j] \to [k]}} \prod_{\substack{\text{nondegenerate} \\ [i_0 \ldots i_j] \in K_j}} X_j \right)$$

In particular, $\operatorname{Hom}(\Delta^k, X) \cong X_k$, while $\operatorname{Hom}(\partial\Delta^k, X) = M_k(X)$ is called the kth **matching space** of $X_\bullet$.

For example, $M_1(X) \cong X_0 \times X_0$.

Reconstructing a groupoid from its nerve, continued

Grothendieck's characterization of nerves of groupoids may be phrased in the following way.

Theorem

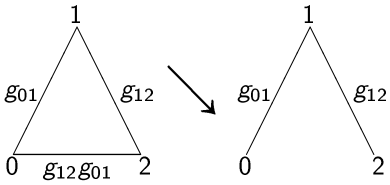
A simplicial set $X_\bullet$ is isomorphic to the nerve of a groupoid if and only if for each $k > 1$ and each $0 \leqslant i \leqslant k$, the natural maps

$$X_k \cong \mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\Lambda^k_i, X)$$

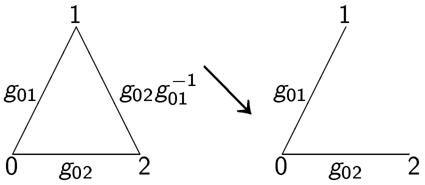
are isomorphisms.
It is isomorphic to the nerve of a group if in addition X_0 has a single element.

In fact, it suffices to assume the condition for $k = 2$ and $k = 3$: the higher cases are then automatically satisfied.

To define the multiplication table of the groupoid, apply this condition to the horn Λ_1^2:



To define the inverse, apply the axiom to the collapse of the 2-simplex at the vertex 0:



The associativity of the product follows by applying the axiom to an elementary collapse of the 3-simplex.

Nerves of Lie groupoids

Definition

The nerve $N_k\mathcal{G} \in s\mathcal{M}$ of a Lie groupoid $\mathcal{G}$ is the simplicial manifold

$$N_k\mathcal{G} = \mathrm{Fun}([k], \mathcal{G}).$$

Note that for $k > 1$, there is something to be checked here: that the space of all functors from $[k]$ to $\mathcal{G}$ is a manifold. This is where we need that the source and target maps are submersions.

Theorem

A simplicial manifold $X_\bullet$ is isomorphic to the nerve of a Lie groupoid if and only if for each $0 \leqslant i \leqslant k$, the natural maps

$$X_k \cong \mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\Lambda^k_i, X)$$

are surjective submersions for $k = 1$, and diffeomorphisms for $k > 1$. It is isomorphic to the nerve of a Lie group if in addition X_0 consists of a single point.

Nerves of Lie groupoids: examples

- The nerve of Lie groupoid associated to a surjective submersion $U \to X$ has the following form:

$$N_n(U/X) = \underbrace{U \times_X \cdots \times_X U}_{n+1 \text{ factors}}$$

  This is called the Čech nerve of the submersion.

- The nerve of the Lie groupoid associated to a group action has the following form:

$$N_n[M/G] = M \times G^n.$$

  This simplicial manifold represents the classifying stack of the action.

Simplicial identities

We will need the following morphisms in Δ.

- the coface $d_i : [n-1] \to [n]$, $0 \leqslant i \leqslant n$, is the monotone map

$$d_i(j) = \begin{cases} j, & j < i \\ j+1, & j \geqslant i \end{cases}$$

- the codegeneracy $s_i : [n+1] \to [n]$, $0 \leqslant i \leqslant n$, is the monotone map

$$s_i(j) = \begin{cases} j, & j \leqslant i \\ j-1, & j > i \end{cases}$$

The induced morphisms for a simplicial object $X_\bullet$ are, respectively,

$$\partial_i : X_n \to X_{n-1}, \qquad \sigma_i : X_n \to X_{n+1}.$$

Together, the cofaces and codegeneracies generate the category Δ, with the relations

- $d_i \circ d_j = d_j \circ d_{i-1}$ if $j < i$
- $s_i \circ d_j = d_j \circ s_{i-1}$ if $j < i$
- $s_i \circ d_i = s_i \circ d_{i+1} = 1$
- $s_i \circ d_j = d_{j-1} \circ s_i$ if $j > i + 1$
- $s_i \circ s_j = s_j \circ s_{i+1}$ if $j \leqslant i$

In Section 2 of the paper

Samuel Eilenberg and Saunders Maclane: On the groups $H(\Pi, n)$, I
Ann. Math. (2) **58** (1953), 55–106

it is proved that every morphism $[k] \to [n]$ of Δ factors in a canonical fashion

$$d_{i_1} \circ \cdots \circ d_{i_p} \circ s_{j_1} \circ \cdots \circ s_{j_q},$$

where $0 \leqslant i_1 < \cdots < i_p < k$ and $n - p + q \geqslant j_1 > \cdots > j_q \geqslant 0$.

Cosimplicial cochain complexes

The de Rham functor $M \mapsto \Omega^*(M)$ is a contravariant functor from differentiable manifolds to cochain complexes over $\mathbb{C}$. If $X_\bullet$ is a simplicial manifold, we see that $\Omega^*(X_\bullet)$ is a **covariant** functor from Δ to cochain complexes. Such a gadget is called a **cosimplicial cochain complex** and denoted $V^\bullet$.

Following Eilenberg and Maclane, we introduce the normalized complex $C^{**}(V)$ of a cosimplicial cochain complex $V^{\bullet *}$. This is the double complex

$$C^{pq}(V) = \begin{cases} V^{0q}, & p = 0 \\ \displaystyle\bigcap_{i=0}^{p-1} \ker\big(s_i : V^{pq} \to V^{p-1,q}\big), & p > 0 \end{cases}$$

The differential on the normalized cochains

The **differential**

$$\delta : C^{pq}(V) \to C^{p+1,q}(V)$$

is the restriction of the alternating sum

$$\delta = \sum_{j=0}^{p+1} (-1)^j \, d_j : V^{pq} \to V^{p+1,q}$$

to $C^{pq}(V) \subset V^{pq}$.

We must check that δ is well-defined, namely, that $s_i\delta = 0$ on $C^{pq}(V)$:

$$\begin{aligned} s_i\delta &= s_i\big(d_0 - d_1 + d_2 + \cdots + (-1)^{p+1} d_{p+1}\big) \\ &= \big(d_0 - d_1 + d_2 + \cdots + (-1)^{i-1} d_{i-1}\big) s_{i-1} \\ &\quad - \big((-1)^{i+1} d_{i+1} + \cdots + (-1)^p d_p\big) s_i \end{aligned}$$

which vanishes on $C^{pq}(V)$.

Furthermore, we have $\delta \circ \delta = 0$:

$$\begin{aligned}
\delta \circ \delta &= \sum_{i=0}^{p+2} \sum_{j=0}^{p+1} (-1)^{i+j} \, d_i d_j \\
&= \sum_{i=0}^{j} \sum_{j=0}^{p+1} (-1)^{i+j} \, d_i d_j + \sum_{i=j+1}^{p+2} \sum_{j=0}^{p+1} (-1)^{i+j} \, d_i d_j \\
&= \sum_{i=0}^{j} \sum_{j=0}^{p+1} (-1)^{i+j} \, d_i d_j + \sum_{i=j+1}^{p+2} \sum_{j=0}^{p+1} (-1)^{i+j} \, d_j d_{i-1} \\
&= 0
\end{aligned}$$

It is clear that δ commutes with the de Rham differential d.

In this way, we see that if $X_\bullet$ is a simplicial manifold, then

$$C^{pq}\big(\Omega^*(X_\bullet)\big)$$

is a double complex, with differentials δ and the de Rham differential d.

Definition

The **de Rham complex** $\Omega^*(X_\bullet)$ of a simplicial manifold $X_\bullet$ is the total complex of the double complex $C^{**}(\Omega^*(X_\bullet))$. Denote its differential by D, and its cohomology by $H^*(X_\bullet)$.

More explicitly, a k-form $\alpha \in \Omega^k(X_\bullet)$ is a collection of differential forms

$$(\alpha_0, \dots, \alpha_k) \in \Omega^k(X_0) \oplus \cdots \oplus \Omega^{k-p}(X_p) \oplus \cdots \oplus \Omega^0(X_k),$$

such that $s_i\alpha_p = 0 \in \Omega^{k-p}(X_{p-1})$ for all $0 \leqslant i < p$.

The differential $D = \delta + (-1)^p\, d$ is given by the formula

$$(D\alpha)_p = \delta\alpha_{p-1} + (-1)^p\, d\alpha_p.$$

The filtration on the de Rham complex

We need a certain filtration on $\Omega^*(X_\bullet)$ induced by a natural filtration ("filtration bête") on the de Rham complex:

$$F^i\Omega^k(X_\bullet) = \begin{cases} 0, & k < i \\ \Omega^k(X_\bullet), & k \geqslant i \end{cases}$$

This filtration induces a filtration $F^i\Omega^*(X_\bullet)$ on the de Rham complex $\Omega^*(X_\bullet)$. A k-form $\alpha \in F^i\Omega^k(X_\bullet)$ is a collection of differential forms

$$(\alpha_0, \dots, \alpha_{k-i}) \in \Omega^k(X_0) \oplus \cdots \oplus \Omega^{k-p}(X_p) \oplus \cdots \oplus \Omega^i(X_{k-i}),$$

such that $s_i\alpha_p = 0 \in \Omega^{k-p}(X_{p-1})$ for all $0 \leqslant i < p$. In particular, $F^i\Omega^k(X_\bullet)$ vanishes for $k < i$.

The sheaf $F^i\Omega^*(M)$ is a fine resolution of the sheaf of closed i-forms. For this reason, it is proposed in the paper

Tony Pantev, Bertrand Töen, Michel Vaquié and Gabriele Vezzosi
Shifted symplectic structures, Publ. Math. IHES (2013), 271–328

to call $F^i\Omega^*(X_\bullet)$ the complex of closed i-forms on the simplicial manifold $X_\bullet$.

The cohomology of the complex $F^i\Omega^*(X_\bullet)$ is unchanged if $X_\bullet$ is replaced by another simplicial manifold representing the same stack: this is a consequence of **cohomological descent**, which we will come to later in these lectures.

Examples

- If $X_\bullet$ is a zero-dimensional simplicial manifold (or equivalently, a simplicial set), then $\Omega^*(X_\bullet)$ may be identified with the complex $C^*(X_\bullet, \mathbb{C})$ of normalized simplical cochains of $X_\bullet$.
- If $X_\bullet = M$ for all $k \geqslant 0$ and all of the maps d_i and s_i are the identity (a *constant* simplicial manifold), then $\Omega^*(X_\bullet) \cong \Omega^*(M)$.
- Let $\mathcal{G}$ be a symplectic groupoid. This is a Lie groupoid together with a symplectic form $\omega \in \Omega^2(\mathcal{G}_1)$ on the manifold of morphisms $\mathcal{G}_1$ such that $\delta\omega = 0$. In other words, ω is a closed form in $F^2\Omega^3(\mathcal{G})$.

The product on the de Rham complex

There is a product on $\Omega^*(X_\bullet)$, defined using the Alexander-Whitney cup-product:

$$(\alpha \cup \beta)_p = \sum_{q=0}^{p} (d_{q+1})^q \alpha_{p-q} \wedge (d_0)^{p-q} \beta_q .$$

This product is not graded commutative, but induces a graded commutative product on $H^*(X_\bullet)$, which agrees with the cup-product on $H^*(|X_\bullet|, \mathbb{C})$ when $X_\bullet$ is good.

By a (more complicated) construction explained in

Xi Zhi Cheng and E.G.: Transferring homotopy commutative algebraic structures. J. Pure Appl. Alg. **212** (2008), 2535–2542

the de Rham complex $\Omega^*(X_\bullet)$ is a C_∞-algebra, that is, a commutative A_∞-algebra. The product in this structure is

$$\frac{1}{2}\big(\alpha \cup \beta + (-1)^{|\alpha|\,|\beta|}\, \beta \cup \alpha\big).$$

Shulman's construction

Let G be a compact group, and let $N_\bullet G$ be the nerve of G.

In his Berkeley thesis (1972), Shulman gives an approach to Chern-Weil theory using the de Rham complex of $N_\bullet G$. Recall that in Chern-Weil theory, we consider the ring $I(G)$ of G-invariant polynomials on the Lie algebra $\mathfrak{g}$.

For example, if G is the unitary group $\mathrm{U}(N)$, then this is a polynomial algebra, generated by the power sums

$$p_j = \mathrm{Tr}(A^j), \quad 1 \leqslant j \leqslant N,$$

or alternatively, by the elementary symmetric functions

$$\det(I + tA) = 1 + \sum_{j=1}^{N} t^j \, c_j .$$

A simplicial G-principle bundle on $N_n G$

The nth space $N_n G$ of the nerve of G is diffeomorphic to G^n. We consider the classifying G-principle bundle $E_\bullet G$ over $N_\bullet G$: this is the simplicial G-principle bundle with

$$E_n G = G^{n+1}.$$

The right action of G is the diagonal one:

$$(h_0, \dots, h_n) \cdot g = (h_0 g, \dots, h_n g).$$

The projection from $E_n G$ to $N_n G$ is given by the explicit formula

$$(h_0, \dots, h_n) \mapsto (g_1, \dots, g_n) = (h_1 h_0^{-1}, \dots, h_n h_{n-1}^{-1}).$$

The simplicial maps are

$$\delta_j(h_0, \dots, h_n) = (h_0, \dots, h_{j-1}, h_{j+1}, \dots, h_n)$$
$$\sigma_j(h_0, \dots, h_n) = (h_0, \dots, h_j, h_j, \dots, h_n)$$

Let $\omega \in \Omega^1(G, \mathfrak{g})$ be the tautological left-invariant one-form on G with values in the Lie algebra $\mathfrak{g}$ of G: $\omega(1)$ is the identity map of $\mathfrak{g}$.

For example, for the unitary group $G = \mathrm{U}(N)$,

$$\omega = h^{-1} dh.$$

The differential form ω satisfies the **Maurer-Cartan equation**

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

For $G = \mathrm{U}(N)$, this may be written

$$d\omega + \omega^2 = 0.$$

Let $\omega_j = \omega(h_j) \in \Omega^1(E_n G, \mathfrak{g})$, $0 \leqslant j \leqslant n$, be the pullback of ω by the jth projection from $E_n G$ to G. Each of these forms defines a connection one-form for the principle bundle $E_n G \to N_n G$.

The geometric n-simplex

Let Δ^n be the geometric n-simplex: this is the convex hull of the basis vectors $\{e_i \mid 0 \leqslant i \leqslant n\} \subset \mathbb{R}^{n+1}$. A point of Δ^n is a sequence $(t_0, \ldots, t_n) \in [0,1]^{n+1}$ such that

$$t_0 + \cdots + t_n = 1.$$

We consider the product of the principle bundle $E_n G \to N_n G$ with the geometric n-simplex Δ^n. On this product, we may form the connection one-form

$$\begin{aligned} \omega &= \sum_{j=0}^{n} t_j \, \omega_j \\ &= \sum_{j=0}^{n} t_j \, h_j^{-1} dh_j \qquad \text{for } G = \mathrm{U}(N) \end{aligned}$$

by taking the affine combination of the connection forms ω_j.

This connection form is certainly not flat: in fact, its curvature equals

$$
\begin{aligned}
F_n &= d\omega + \frac{1}{2}[\omega, \omega] \\
&= \sum_{j=0}^{n} dt_j \wedge \omega_j + \sum_{j=0}^{n} \Big(t_j \, d\omega_j + \frac{1}{2} \, t_j^2 \, [\omega_j, \omega_j] \Big) \\
&\qquad + \sum_{0 \leqslant i < j \leqslant n} t_i t_j \, [\omega_i, \omega_j] \in \Omega^2(E_n G, \mathfrak{g}) \\
&= \sum_{j=0}^{n} dt_j \wedge \omega_j + \frac{1}{2} \sum_{j=0}^{n} (t_j^2 - t_j)[\omega_j, \omega_j] \\
&\qquad + \sum_{0 \leqslant i < j \leqslant n} t_i t_j \, [\omega_i, \omega_j] \in \Omega^2(E_n G, \mathfrak{g}) \\
&= \sum_{j=1}^{n} dt_j \wedge (\omega_j - \omega_0) + \frac{1}{2} \sum_{j=1}^{n} (t_j^2 - t_j)[\omega_j - \omega_0, \omega_j - \omega_0] \\
&\qquad + \sum_{1 \leqslant i < j \leqslant n} t_i t_j \, [\omega_i - \omega_0, \omega_j - \omega_0] \in \Omega^2(E_n G, \mathfrak{g})
\end{aligned}
$$

Suppose that $P \in I^k(G)$ is an invariant polynomial of degree k. By the main theorem of Chern-Weil theory, the differential form

$$P\big((-2\pi i)^{-1} F_n\big) \in \Omega^{2k}(\mathbb{A}^n \times E_n G)$$

is closed and basic with respect to the right G-action, and thus descends to a closed $2k$-form on $\mathbb{A}^n \times N_n G$.

Shulman's idea is to generalize the construction of Chern and Simons, by integrating this form over the n-simplex:

$$\Phi_n(P) = \int_{\mathbb{A}^n} P\big((-2\pi i)^{-1} F_n\big).$$

Theorem (Shulman)

1. $\Phi_n(P) = 0$ *for* $n > k$
2. $d\Phi_n(P) = \delta\Phi_{n-1}(P)$
3. $s_j\Phi_n(P) = 0$ *for* $0 \leqslant j \leqslant n$

In summary, we have constructed a closed differential form

$$\Phi(P) = (0, \Phi_1(P), \dots, \Phi_k(P), 0, \dots) \in F^k\Omega^{2k}(N_\bullet G).$$

This construction is linear in P, so we actually have a linear map

$$\Phi : I^k(G) \to F^k\Omega^{2k}(N_\bullet G).$$

Proof of Shulman's theorem

The differential form $P\big((-2\pi i)^{-1}F_n\big)$ has degree at most k in $\{dt_0,\ldots,dt_n\}$, hence its integral over the n-simplex vanishes for $n > k$. Thus $\Phi_n(P) = 0$ for $n > k$.

The second part of Shulman's theorem follows from Stokes's Theorem:

$$d\int_{\mathbb{\Delta}^n} P\big((-2\pi i)^{-1}F_n\big) = (-1)^n\int_{\mathbb{\Delta}^n} dP\big((-2\pi i)^{-1}F_n\big) + \sum_{j=0}^{n}(-1)^j\int_{\partial_j\mathbb{\Delta}^n} P\big((-2\pi i)^{-1}F_n\big).$$

But the first term on the right-hand side vanishes, since $P\big((-2\pi i)^{-1}F_n\big)$ is a Chern-Weil characteristic form.

To prove that $s_j \Phi_n(P) = 0$, we consider the diagram

$$\begin{array}{ccc} \mathbb{\Delta}^{n+1} \times E_n G & \xrightarrow{\alpha} & \mathbb{\Delta}^{n+1} \times E_{n+1} G \\ \Big\downarrow \beta & & \\ \mathbb{\Delta}^{n} \times E_n G & & \end{array}$$

where $\alpha = \mathbb{\Delta}^{n+1} \times \sigma_j$ and $\beta = s_j \times E_n G$.

We have $\alpha^* F_{n+1} = \beta^* F_n \in F^k \Omega^{2k}(\mathbb{\Delta}^{n+1} \times E_n G)$. It follows that

$$\begin{aligned} s_j \Phi_n(P) &= \int_{\mathbb{\Delta}^n} \alpha^* P\big((-2\pi i)^{-1} F_{n+1}\big) \\ &= \int_{\mathbb{\Delta}^{n+1}} \beta^* P\big((-2\pi i)^{-1} F_n\big) \\ &= 0 \end{aligned}$$

This completes the proof of Shulman's theorem.

Problem

In what sense is this map compatible with products?

Optimistically, given $P \in I^j(G)$ and $Q \in I^k$, there is a naturally defined differential form $\Psi(P,Q) \in F^{j+k}\Omega^{2j+2k-1}(N_\bullet G)$ such that

$$\Phi(PQ) = \Phi(P) \cup \Phi(Q) + D\Psi(P,Q).$$

Example: $P = c_1$

For our first example, we consider the invariant polynomial

$$P(A) = \mathrm{Tr}(A) \in I^1(\mathrm{U}(N))$$

on $\mathfrak{u}(N)$. This corresponds to the first Chern class of a vector bundle under the Chern-Weil map (and also to the class p_1, which is the same as the first Chern character class ch_1).

We will use the formula

$$g_1^{-1} dg_1 = h_0(\omega_1 - \omega_0) h_0^{-1}$$

Since $k = 1$, we see that $\Phi_n(P)$ vanishes unless $n = 1$:

$$\begin{aligned} \Phi_1(P) &= -\frac{1}{2\pi i} \int_0^1 dt_1 \ \mathrm{Tr}\big(\omega_1 - \omega_0\big) \\ &= -\frac{1}{2\pi i} \mathrm{Tr}\big(\omega_1 - \omega_0\big) \\ &= -\frac{1}{2\pi i} \mathrm{Tr}\big(g_1^{-1} dg_1\big) \end{aligned}$$

The equation $\delta\Phi_1(P) = 0$ follows from Shulman's theorem. As we mentioned at the end of the first lecture, this equation is exactly the equation

$$d \log \det(g_1 g_2) = d \log \det(g_1) + d \log \det(g_2)$$

up to a factor of $(-2\pi i)^{-1}$.

To complete the proof of the existence of the determinant, we should show that this closed one-form on $\mathrm{U}(N)$ has integral periods. This follows from the theory of Chern classes.

It is easily checked directly in this case, since $H_1(\mathrm{U}(N), \mathbb{Z})$ has rank one, and is spanned by the one-cycle $\mathrm{U}(1) \to \mathrm{U}(N)$ associated to any primitive cocharacter, for example

$$z \mapsto \mathrm{diag}(z, 1, \ldots, 1).$$

Example: $P = c_2$

We now turn to the second Chern class c_2, which corresponds to the invariant polynomial

$$P(A) = -\frac{1}{2}\big(\mathrm{Tr}(A^2) - \mathrm{Tr}(A)^2\big) \in I^2(\mathrm{U}(N))$$

on $\mathfrak{u}(N)$. This is the second elementary symmetric function in the eigenvalues of A:

$$P(A) = \sum_{1 \leqslant i < j \leqslant N} \lambda_i \lambda_j.$$

We have

$$P((-2\pi i)^{-1} F_n) = \frac{1}{8\pi^2}\,\mathrm{Tr}(F_n^2) - \frac{1}{8\pi^2}\,\mathrm{Tr}(F_n)^2$$

Since $k = 2$, we see that $\Phi_n(P)$ vanishes unless $n = 1$ or 2.

For $n = 1$, the term $\operatorname{Tr}(F_1)^2$ vanishes, and we have

$$\begin{aligned}\Phi_1(c_2) &= \frac{1}{8\pi^2}\int_0^1 dt_1\,(t_1^2 - t_1)\operatorname{Tr}\big((\omega_1-\omega_0)^3\big)\\ &= -\frac{1}{48\pi^2}\operatorname{Tr}\big((\omega_1-\omega_0)^3\big)\\ &= -\frac{1}{48\pi^2}\operatorname{Tr}\big((g_1^{-1}dg_1)^3\big)\end{aligned}$$

This differential form is closed, and may be checked to represent a generator of $H^3(\mathrm{U}(N),\mathbb{Z})$, using the fact that any homomorphism

$$\mathrm{SU}(2) \hookrightarrow \mathrm{U}(N)$$

embedding $\mathrm{SU}(2)$ in $\mathrm{U}(N)$ defines a cycle generating $H_3(\mathrm{U}(N),\mathbb{Z})$.

When $P = c_2$, the differential $\delta\Phi_1(P) \in \Omega^3(\mathrm{U}(N) \times \mathrm{U}(N))$ is nonzero. However, it is exact, and its nonvanishing is measured by the differential form

$$\Phi_2(P) \in \Omega^2(\mathrm{U}(N) \times \mathrm{U}(N)),$$

which satisfies $d\Phi_2(P) = \delta\Phi_1(P)$ and $\delta\Phi_2(P) = 0$. We now turn to the calculation of $\Phi_2(P)$.

We must extract the terms of $P((-2\pi i)^{-1}F_2)$ proportional to $dt_1 \wedge dt_2$: these are

$$-\frac{1}{4\pi^2}\, dt_1 \wedge dt_2 \wedge \Big(\mathrm{Tr}\big((\omega_1 - \omega_0)(\omega_2 - \omega_0)\big) - \mathrm{Tr}(\omega_1 - \omega_0) \wedge \mathrm{Tr}(\omega_2 - \omega_0)\Big)$$

Observe that

$$\begin{aligned}\omega_1 - \omega_0 &= h_0^{-1}\big(g_1^{-1}(dg_1)\big)h_0 \\ &= h_1^{-1}\big((dg_1)g_1^{-1}\big)h_1 \\ \omega_2 - \omega_0 &= h_1^{-1}\big(g_2^{-1}(dg_2)\big)h_1 + (\omega_1 - \omega_0)\end{aligned}$$

Integrating over $\mathbb{A}^2$, we gain an additional factor of $\frac{1}{2}$, obtaining

$$\begin{aligned}\Phi_2(P) &= -\frac{1}{8\pi^2}\Big(\mathrm{Tr}\big((\omega_1-\omega_0)(\omega_2-\omega_0)\big) - \mathrm{Tr}(\omega_1-\omega_0)\wedge\mathrm{Tr}(\omega_2-\omega_0)\Big) \\ &= -\frac{1}{8\pi^2}\Big(\mathrm{Tr}\big(\big((dg_1)g_1^{-1}\big)\big(g_2^{-1}(dg_2)\big)\big) - \mathrm{Tr}(g_1^{-1}dg_1)\wedge\mathrm{Tr}(g_2^{-1}dg_2)\Big)\end{aligned}$$

For the simple group $\mathrm{SU}(N)$, the second term vanishes. Its presence ensures that $\delta\Phi_2(P) = 0$: it plays no rôle in the formula $d\Phi_2(P) = \delta\Phi_1(P)$.

The Polyakov-Wiegmann formula is a lift of this result to differential characters — it may be proved in the same way, with integration over $\mathbb{\Delta}^n$ replaced by the lift of Stokes's theorem to differential characters proved in

K. Gomi and Y. Terashima: A fiber integration formula for the smooth Deligne cohomology. IMRN (2000), 699–708

For further explicit calculations along the lines of the above examples, see

Naoya Suzuki: The Chern character in the simplicial de Rham complex. `arxiv:1306.5949`

Hypercovers of simplicial manifolds

Certain maps of simplicial manifolds should be thought of as equivalences. This is already seen in the definition of manifolds in terms of an atlas: if $U \to X$ is a surjective submersion (and in particular, an open cover), the natural map from the the Čech nerve $N_\bullet(U/X)$ to the constant simplicial manifold X is an equivalence of Lie groupoids.

Definition (Verdier, SGA 4)

A morphism $X_\bullet \to Y_\bullet$ of simplicial schemes is a **hypercover** if the map

$$X_k \to M_k(X) \times_{M_k(Y)} Y_k$$

is a surjective étale morphism for all $k \geqslant 0$.

(Recall that the matching space $M_k(X)$ equals $\mathrm{Hom}(\partial\Delta^k, X)$.)

For $k = 0$, this says that $X_0 \to Y_0$ is surjective and étale.

For $k = 1$, it says that $X_1 \to (X_0 \times X_0) \times_{(Y_0 \times Y_0)} Y_1$ is surjective and étale.

The following lemma is proved in

D. Dugger, S. Hollander, D. C. Isaaksen: Hypercovers and simplicial presheaves. Math. Proc. Cambridge Philos. Soc. **136** (2004), 9–51

Lemma

If $X_\bullet \to Y_\bullet$ is a morphism of simplicial manifolds such that $M_i(X) \times_{M_i(Y)} Y_i$ is a manifold for all $i < k$, and the map

$$X_i \to M_i(X) \times_{M_i(Y)} Y_i$$

is a surjective submersion for all $0 \leqslant i < k$, then $M_k(X) \times_{M_k(Y)} Y_k$ is a manifold.

(Recall that the matching space $M_k(X)$ equals $\mathrm{Hom}(\partial\Delta^k, X)$.)

This lemma allows us to extend Verdier's definition of hypercovers to simplicial manifolds (where fibred products do not exist in general): we simply replace schemes by manifolds, and surjective étale morphisms by surjective submersions.

Hypercovers have many good properties: in particular, they satisfy the four axioms for a topology $\mathcal{P}$ on a category $\mathcal{C}$ that we stated in the first lecture:

1. identity maps are hypercovers;
2. the composition of two hypercovers is a hypercover;
3. pullbacks of hypercovers along simplicial maps exist, and are again hypercovers;
4. if gf and f are hypercovers, then g is a hypercover.

Consider the case of the Čech nerve associated to a surjective submersion $U \to X$. In this case,

$$M_k(N_\bullet(U/X)) \cong \begin{cases} \text{pt}, \\ U \times U, \\ N_k(U/X), \end{cases} \qquad M_k(X) \cong \begin{cases} \text{pt}, & k = 0 \\ X \times X, & k = 1 \\ X, & k > 1 \end{cases}$$

The map $N_\bullet(U/X) \to X$ of simplicial manifolds is a hypercover, since

- $(k = 0)$ $U \to X$ is a surjective submersion;
- $(k = 1)$ $U \times_X U \to (U \times U) \times_{X \times X} X$ is a surjective submersion;
- $(k > 1)$ $N_k(U/X) \to M_k(N_\bullet(U/X))$ is a surjective submersion.

For $k = 0$, this is true by hypothesis, while for $k > 0$, these maps are actually isomorphisms.

Cohomological descent

Cohomological descent was introduced in Section 5.3 of the paper

Pierre Deligne: Théorie de Hodge, III. Publ. Math. IHES **44** (1974), 5–77

A map $X_\bullet \to Y_\bullet$ of simplicial manifolds induces a map of de Rham complexes $\Omega^*(Y_\bullet) \to \Omega^*(X_\bullet)$.

Theorem

If $X_\bullet \to Y_\bullet$ is a hypercover, the induced map of complexes of closed i-forms

$$F^i\Omega^*(Y_\bullet) \to F^i\Omega^*(X_\bullet)$$

is a quasi-isomorphism.

Deligne states the special case of this theorem in which $Y_\bullet$ is a constant simplicial manifold. (The details are in B. Saint-Donat, SGA4, Exp. Vbis.) For the general case, we will follow the appendix of the paper of Dugger, Hollander and Isaksen.

Cohomological descent: Čech nerves

We prove the theorem in a series of steps.

Cohomological descent holds for the hypercover

$$N_\bullet(U/X) \to X$$

associated to an open cover $U \to X$ of a manifold X (that is, U is the disjoint union of open subsets of X which cover X). This result is contained in Weil's proof of the de Rham theorem, and uses a partition of unity for the cover. (See for example §8 of Bott and Tu.)

Our next goal is to prove cohomological descent for the hypercover

$$N_\bullet(U/X) \to X$$

associated to a surjective submersion $U \to X$. By the previous result, we may replace X by an open cover V, and U by the fibred product $U \times_X V$.

Hence, we may as well assume that the submersion $U \to X$ has a differentiable section σ.

Coaugmented cosimplicial objects

The category Δ_+ is obtained from Δ by adjoining an initial object $[-1]$, the empty category.

Definition

An **coaugmented** cosimplicial cochain complex is a covariant functor from Δ_+ to cochain complexes.

Equivalently, a coaugmentation of a cosimplicial cochain complex $V^{\bullet *}$ is a cochain complex $W^* = V^{-1*}$, together with a map of complexes $d_0 : W^* \to V^{0*}$, such that

$$d_0 d_0 = d_1 d_0 : W^* \to V^{1*}$$

It is straightforward to extend the definition of the normalized cochain complex to coaugmented cosimplicial cochain complexes.

If $U \to X$ is a surjective submersion, there is a coaugmented cosimplicial cochain complex $V^{\bullet *}$ whose underlying cosimplicial cochain complex is $F^i\Omega^*(N_\bullet(U/X))$:

$$V^{-1*} = F^i\Omega^*(X),$$

and $d_0 : V^{-1*} \to V^{0*}$ is the pullback $F^i\Omega^*(X) \to F^i\Omega^*(U)$.

We will prove that this complex is contractible, under the additional hypothesis that $U \to X$ has a section $\sigma : X \to U$.

Extra codegeneracies

Definition

An **extra codegeneracy** on a coaugmented cosimplicial vector space $V^\bullet$ is a map $s_{-1} : V^{n+1} \to V^n$, $n \geqslant 0$, such that

$$s_{-1} \circ d_0 = 1 : V^n \to V^n, \qquad n \geqslant -1,$$
$$s_{i-1} \circ s_{-1} = s_{-1} \circ s_i : V^{n+1} \to V^{n-1}, \qquad 0 \leqslant i \leqslant n.$$

The operators s_{-1} induce a contraction of the normalized cochain complex $C^*(V^\bullet)$:

$$s_{-1} \circ \delta + \delta \circ s_{-1} = 1.$$

Cohomological descent for Čech nerves

In the case of the coaugmented cosimplicial cochain complex associated to a surjective submersion $U \to X$ with section $\sigma : X \to U$, the extra degeneracy is defined by pullback along the map

$$\sigma_{-1} : N_n(U/X) \cong N_n(U/X) \times_X X \xrightarrow{N_n(U/X) \times_X \sigma} N_n(U/X) \times_X U \cong N_{n+1}(U/X)$$

Coskeleta

The proof of cohomological descent proceeds by decomposing a hypercover using the coskeleton functors.

The m-skeleton $\operatorname{sk}_m \Delta^n$ of the n-simplex is the union of all of its m-dimensional simplices together with their degeneracies. That is, the simplex $[i_0 \dots i_k]$ is in the m-skeleton if and only if the set $\{i_0, \dots, i_k\}$ has cardinality at most m. For example, $\operatorname{sk}_{n-1} \Delta^n = \partial\Delta^n$.

If $X_\bullet$ is a simplicial set, its m-**coskeleton** is the simplicial set

$$(\operatorname{cosk}_m X)_n = \operatorname{Hom}(\operatorname{sk}_m \Delta^n, X).$$

If $f : X_\bullet \to Y_\bullet$ is a map of simplicial sets, the **relative m-coskeleton** is the simplicial set

$$(\operatorname{cosk}_m(X/Y))_n = \operatorname{Hom}(\operatorname{sk}_m \Delta^n, X) \times_{\operatorname{Hom}(\operatorname{sk}_m \Delta^n, Y)} Y_n.$$

In this way, we obtain a tower of maps

$$X_\bullet \to \cdots \to \operatorname{cosk}_1(X/Y) \to \operatorname{cosk}_0(X/Y) \to \operatorname{cosk}_{-1}(X/Y) \cong Y_\bullet.$$

If f is a hypercover of simplicial sets, then each of the maps in this diagram is a hypercover.

Thus, to show that the cohomology of simplicial sets is invariant under forming hypercovers, it suffices to show that

$$H^*(\operatorname{cosk}_{m-1}(X/Y)) \cong H^*(\operatorname{cosk}_m(X/Y))$$

for all $m \geqslant 0$, since $H^k(\operatorname{cosk}_m(X/Y)) \cong H^k(X_\bullet)$ as soon as $m > k$.

Coskeleta for hypercovers

Lemma

If $f : X_\bullet \to Y_\bullet$ is a hypercover of simplicial manifolds, $\mathrm{cosk}_m(X/Y)$ is a simplicial manifold for all $m \geqslant 0$, and the induced map of simplicial manifolds

$$\mathrm{cosk}_m(X/Y) \longrightarrow \mathrm{cosk}_{m-1}(X/Y)$$

is a hypercover.

The first part of the lemma is a special case of the following general result.

Lemma

If $f : X_\bullet \to Y_\bullet$ is a hypercover of simplicial manifolds and $K \subset \Delta^n$ is a simplicial subset of the n-simplex, then

$$\mathrm{Hom}(K, X) \times_{\mathrm{Hom}(K,Y)} Y_n$$

is a simplicial manifold.

This is proved by replacing K by its k-skeleton $\mathrm{sk}_k K$ and using induction on k. For $k = -1$, $\mathrm{sk}_{-1} K$ is the empty simplicial set, and

$$\mathrm{Hom}(\mathrm{sk}_{-1} K, X) \times_{\mathrm{Hom}(\mathrm{sk}_{-1} K, Y)} Y_n \cong Y_n$$

is a manifold.

The induction step follows from the pullback diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{sk}_k K, X) \times_{\mathrm{Hom}(\mathrm{sk}_k K, Y)} Y_n & \longrightarrow & \displaystyle\prod_{\substack{[i_0 \dots i_k] \in K \\ 0 \leqslant i_0 < \dots < i_k \leqslant n}} X_k \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathrm{sk}_{k-1} K, X) \times_{\mathrm{Hom}(\mathrm{sk}_{k-1} K, Y)} Y_n & \longrightarrow & \displaystyle\prod_{\substack{[i_0 \dots i_k] \in K \\ 0 \leqslant i_0 < \dots < i_k \leqslant n}} M_k(X) \times_{M_k(Y)} Y_k
\end{array}$$

in which the right vertical arrow, and hence the left vertical arrow, are surjective submersions.

To prove that the simplicial map

$$\operatorname{cosk}_m(X/Y) \to \operatorname{cosk}_{m-1}(X/Y)$$

is a hypercover, we must show that the maps

$$\operatorname{cosk}_m(X/Y)_n \to M_n(\operatorname{cosk}_m(X/Y)) \times_{M_n(\operatorname{cosk}_{m-1}(X/Y))} \operatorname{cosk}_{m-1}(X/Y)_n$$

are surjective submersions.

In fact, these maps are isomorphisms for $n \neq m$, while for $n = m$, this map is the surjective submersion

$$X_m \to M_m(X) \times_{M_m(Y)} Y_m.$$

This follows by the natural isomorphisms

$$\begin{aligned} M_n(\operatorname{cosk}_m(X/Y)) &\times_{M_n(\operatorname{cosk}_{m-1}(X/Y))} \operatorname{cosk}_{m-1}(X/Y)_n \\ &\cong \operatorname{Hom}(\operatorname{sk}_m \partial\Delta^n \cup \operatorname{sk}_{m-1} \Delta^n, X) \times_{\operatorname{Hom}(\operatorname{sk}_m \partial\Delta^n \cup \operatorname{sk}_{m-1} \Delta^n, Y)} Y_n \end{aligned}$$

The following lemma is an easy exercise.

Lemma

If $X_\bullet \to Y_\bullet$ is a hypercover, the maps $X_n \to Y_n$ are surjective submersions.

Given a hypercover $X_\bullet \to Y_\bullet$, consider the bisimplicial manifold

$$Z_{jk} = N_j(X_k/Y_k).$$

Applying the de Rham functor $F^i\Omega^*(M)$ to the bisimplicial manifold $Z_{\bullet\bullet}$, we obtain a **bicosimplicial** cochain complex

$$V^{\bullet\bullet*} = F^i\Omega^*(Z_{\bullet\bullet})$$

Denote the coface maps by

$$d_j : V^{\bullet\bullet*} \to V^{\bullet+1\bullet*}$$
$$\bar{d}_j : V^{\bullet\bullet*} \to V^{\bullet+1\bullet*}$$

and the codegeneracy maps by

$$s_j : V^{\bullet\bullet*} \to V^{\bullet-1\bullet*}$$
$$\bar{s}_j : V^{\bullet\bullet*} \to V^{\bullet-1\bullet*}$$

The normalization of a bicosimplicial cochain complex is defined in a similar way to the normalization of a cosimplicial cochain complex:

$$N^{j,k}(V^{\bullet\bullet}) = \left\{ v \in V^{j,k} \;\middle|\; \begin{matrix} s_p v = 0 \in V^{j-1,k},\ 0 \leqslant p \leqslant j-1 \\ \bar{s}_q v = 0 \in V^{j,k-1},\ 0 \leqslant q \leqslant k-1 \end{matrix} \right\}$$

This is equivalent to normalizing succesively in each direction.

On the normalization, we have the differentials

$$\delta = \sum_{p=0}^{j} (-1)^p \, d_p : N^{j,k}(V^{\bullet\bullet}) \to N^{j+1,k}(V^{\bullet\bullet})$$

$$\bar{\delta} = \sum_{q=0}^{k} (-1)^q \, \bar{d}_q : N^{j,k}(V^{\bullet\bullet}) \to N^{j,k+1}(V^{\bullet\bullet})$$

On the subspace $N^{j,k}(F^i\Omega^*(Z_{\bullet\bullet})) \subset F^i\Omega^*(Z)$ of the total complex of the triple complex $N^{**}(F^i\Omega^*(Z_{\bullet\bullet}))$, we have the total differential

$$D = \delta_1 + (-1)^j \, \delta_2 + (-1)^{j+k} \, d$$

Cohomological descent for Čech covers yields the quasi-isomorphism

$$F^i\Omega^*(Z) \simeq F^i\Omega^*(Y).$$

To complete the proof of cohomological descent, we will use the Eilenberg-Zilber theorem. If $V^{\bullet\bullet}$ is a bicosimplicial vector space, let $\widehat{V}^\bullet$ be the diagonal cosimplicial vector space of $V^{\bullet\bullet}$:

$$\widehat{V}^n = V^{nn},$$

with cofaces $\hat{d} = d_j \circ \bar{d}_j$ and codegeneracies $\hat{s}_j = s_j \circ \bar{s}_j$.

There is a map of graded vector spaces f between the normalizations of the bicosimplicial vector space $V^{\bullet\bullet}$ and the cosimplicial vector space $\widehat{V}^\bullet$: if $v \in N^{j,k}(V^{\bullet\bullet}) \subset V^{j,k}$, then

$$f(v) = (d_{j+1})^k \circ (\bar{d}_0)^j v \in \widehat{V}^{j+k}.$$

It may be checked that $f(v)$ is normalized, so that f induces a map

$$f : N^*(V^{\bullet\bullet}) \to N^*(\widehat{V}^{\bullet}).$$

Furthermore, f is a map of cochain complexes.

Theorem (Eilenberg-Zilber)

The map f is a quasi-isomorphism of cochain complexes.

The theorem is proved in Section 2 of

Samuel Eilenberg and Saunders Maclane: On the groups $H(\Pi, n)$, II
Ann. Math. (2) **60** (1954), 49–139

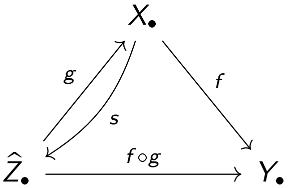
They construct an explicit quasi-inverse g for f, and homotopies $fg \sim 1$ and $gf \sim 1$. (The map g is a sum over shuffles. It turns out that $gf = 1$.)

The proof of cohomological descent is completed by the following lemma of Dugger, Hollander and Isaksen.

Lemma

If $f : X_\bullet \to Y_\bullet$ is a hypercover such that $X_\bullet = \mathrm{cosk}_m(X/Y)$ and $Y_\bullet = \mathrm{cosk}_{m-1}(X/Y)$, then there is a natural quasi-isomorphism between the complexes $F^i\Omega^*(\hat{Z})$ and $F^i\Omega^*(X)$, where $\hat{Z}_\bullet$ is the diagonal simplicial manifold of $Z_{\bullet\bullet}$.

In order to prove this result, we will construct a diagram of simplicial manifolds



such that $g \circ s = 1$. Since pullback along $f \circ g$ is a quasi-isomorphism between $F^i\Omega^*(\hat{Z})$ and $F^i\Omega^*(Y)$, this proves the lemma.

The simplicial map s takes $x \in Z_{0n} \cong X_n$ to $(\sigma_0)^n x \in \hat{Z}_n \cong Z_{nn}$.

Since $X_\bullet \cong \operatorname{cosk}_m(X/Y)$, the simplicial map g is determined by its restriction to $\hat{Z}_n$, $0 \leqslant n \leqslant m$. For $0 \leqslant n < m$, we have $\hat{Z}_n \cong X_n \cong Y_n$, and g_n is the identity. For $n = m$, $\hat{Z}_m \cong N_m(X_m/Y_m)$, α_m is the inclusion of the diagonal copy of X_m in $\hat{Z}_m$, and we may define

$$g_m = (\partial_0)^m : \hat{Z}_m \cong Z_{mm} \to X_m \cong Z_{0m}$$

to be the projection to the last factor. It is not hard to check that this defines a map g on the m-truncation of $\hat{Z}_\bullet$, which thus extends to the desired simplicial map g.

Lie ∞-groupoids

The following class of simplicial manifolds are a natural generalization of Lie groupoids: notice the similarity to the definition of Kan complexes.

Definition

A simplicial manifold $X_\bullet$ is a **Lie ∞-groupoid** if and only if for each $0 \leqslant i \leqslant k$, the natural maps

$$X_k \cong \mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\Lambda^k_i, X)$$

are surjective submersions for $k > 0$.
It is a **Lie ∞-group** if in addition X_0 consists of a single point.

Lie ∞-groupoids form a subcategory of the category of simplicial manifolds.

Lie n-groupoids

We have further subcategories, defined in imitation of Duskin's definition of n-groupoids.

Definition

A simplicial manifold $X_\bullet$ is a **Lie n-groupoid** if and only if for each $0 \leqslant i \leqslant k$, the natural maps

$$X_k \cong \mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\Lambda^k_i, X)$$

are surjective submersions for $k > 0$, and a diffeomorphism for $k > n$.
It is a **Lie n-group** if in addition X_0 consists of a single point.

When $n = 1$, we recover the categories of Lie groupoids and Lie groups respectively.

Differentiable ∞-stacks

The relationship between hypercovers and Lie ∞-groupoids is as follows.

Lemma

If $X_\bullet \to Y_\bullet$ is a hypercover and either $X_\bullet$ or $Y_\bullet$ are Lie ∞-groupoids, then both $X_\bullet$ and $Y_\bullet$ are.

Two Lie ∞-groupoids are **equivalent** if there is a diagram of hypercovers

$$X_\bullet \twoheadleftarrow P_\bullet \twoheadrightarrow Y_\bullet$$

Definition

Differentiable ∞-stacks are equivalence classes of Lie ∞-groupoids.

The characterization of equivalences

Theorem (Behrend-Getzler, 2012)

A morphism $X_\bullet \to Y_\bullet$ of Lie ∞-groupoids is an ***equivalence*** *if and only if the map*

$$X_k \times_{Y_k} Y_{k+1} \to M_k(X) \times_{M_k(Y)} \mathrm{Hom}(\Lambda^{k+1}_{k+1}, Y)$$

is a surjective submersion for all $k \geqslant 0$.
If $X_\bullet$ and $Y_\bullet$ are Lie n-groupoids, it suffices to consider $0 \leqslant k \leqslant n+1$.

The case $k = 0$ of this condition is illuminating:

The map $X_0 \times_{Y_0} Y_1 \to Y_0$ is a surjective submersion

This is analogous to the condition for maps of topological spaces $f : X \to Y$ that the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ be surjective, in other words, that f be 0-connected.

Differentiable n-stacks

Definition

Differentiable n-stacks are equivalence classes of Lie n-groupoids.

In the case where $X_\bullet$ and $Y_\bullet$ are 1-groupoids (nerves of Lie groupoids), this is the same thing as Morita equivalence.

Theorem (Wolfson, 2013)

If $X_\bullet$ and $Y_\bullet$ are Lie n-groupoids which are equivalent as Lie ∞-groupoids, by hypercovers $P_\bullet \to X_\bullet$ and $P_\bullet \to Y_\bullet$, then

- $\operatorname{cosk}_n(P/X)$ *is a Lie n-groupoid.*
- *there is an induced hypercover* $\operatorname{cosk}_n(P/X) \to Y_\bullet$*, and hence a diagram of hypercovers of Lie n-groupoids*

$$X_\bullet \xleftarrow{\sim} \operatorname{cosk}_n(P/X) \xrightarrow{\sim} Y_\bullet$$

De Rham's theorem for Lie ∞-groupoids

The **geometric realization** of a simplicial manifold $X_\bullet$ is the topological space

$$|X_\bullet| = \coprod_{n=0}^{\infty} X_n \times \mathbb{\Delta}^n / \sim$$

where for $x \in X_n$,

$$(\partial_i x, t_0, \ldots, t_{n-1}) \sim (x, t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$
$$(\sigma_i x, t_0, \ldots, t_{n+1}) \sim (x, t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n-1})$$

The following theorem is the generalization to Lie ∞-groupoids of the usual de Rham theorem (which is, of course, needed in the proof).

Theorem

If $X_\bullet$ is a Lie ∞-groupoid, there is a natural isomorphism (of graded commutative algebras) $H^*(X_\bullet) \cong H^*(|X_\bullet|, \mathbb{C})$.

Remarks on the proof of the de Rham theorem

The proof of de Rham's theorem is essentially contained in Appendix A of

G. B. Segal: Categories and cohomology theories
Topology **13** (1974), 293–312

Segal first proves that the de Rham cohomology of $X_\bullet$ is isomorphic to the singular cohomology of the **fat** geometric realization

$$\|X_\bullet\| = \coprod_{n=0}^{\infty} X_n \times \mathbb{\Delta}^n / \approx$$

where for $x \in X_n$,

$$(\partial_i x, t_0, \ldots, t_{n-1}) \approx (x, t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

(In other words, we don't quotient by degeneracies.)

Segal proves that if $X_\bullet$ is a simplicial manifold for which the degeneracies $\sigma_i : X_{n-1} \to X_n$ are closed embeddings (such simplicial manifolds are called **good**), then there is a weak homotopy equivalence

$$|X_\bullet| \simeq \|X_\bullet\|$$

The simplicial manifold underlying a Lie ∞-groupoid satisfies this criterion, since $\sigma_i : X_{n-1} \to X_n$ is a section of the submersion $\partial_i : X_n \to X_{n-1}$.
(I owe this observation to D. Roytenberg.)

Symplectic n-stacks

We would like to give a general definition of symplectic n-stacks, following the ideas of Pantev et al. (As mentioned in the introduction, they define the much more general notion of *derived* symplectic n-stacks.)

Definition

A pre-symplectic n-groupoid is a Lie n-groupoid $X_\bullet$, together with a differential form $\Omega \in F^2\Omega^{n+2}(X_\bullet)$ such that $D\Omega = 0$.
A hypercover $f : (X_\bullet, \Omega) \to (Y_\bullet, \Psi)$ of pre-symplectic n-groupoids is a hypercover $f : X_\bullet \to Y_\bullet$ of Lie n-groupoids together with a differential form $\alpha \in F^2\Omega^{n+1}(X)$ such that $f^*\Psi = \Omega + D\alpha$.

A pre-symplectic differentiable n-stack is an equivalence class of pre-symplectic n-groupoids. This definition is justified by the theorem of cohomological descent for $F^2\Omega^*(X)$: it wouldn't really make sense if this theorem were not true.

Non-degeneracy

We recall that the differential form Ω decomposes into components

$$\Omega_k \in \Omega^{n-k+2}(X_k), \quad 0 \leqslant k \leqslant n.$$

In particular, Ω_n is a two-form on X_n satisfying $\delta\Omega_n = 0$: such a differential form is sometimes referred to as **multiplicative**.

In order to be a symplectic n-stack, we must impose an additional condition of non-degeneracy. In order to define this, we recall the definition of the cotangent complex of a Lie n-groupoid.

The tangent complex

Let $X_\bullet$ be a simplicial manifold. The tangent bundles TX_k may be pulled back to the manifold X_0 by the iterated degeneracy map $(\sigma_0)^k$:

$$\mathbb{T}_k = \left((\sigma_0)^k\right)^* TX_k.$$

This yields a simplicial vector bundle $\mathbb{T}_\bullet$ on X_0.

At each point $x \in X_0$, the fibre of $\mathbb{T}_\bullet$ is a simplicial vector space, with normalized chain complex $C_*(\mathbb{T}_x)$. The normalized chain complex of a simplicial vector space is defined in a similar way to the normalized cochain complex of a cosimplicial vector space, but with cokernels replacing kernels: if $V_\bullet$ is a simplicial vector space,

$$C_p(V) = \begin{cases} V_0, & p = 0, \\ V_p \Big/ \bigoplus_{i=0}^{p-1} \operatorname{im}\Big(\sigma_i : V_{p-1} \to V_p\Big), & p > 0. \end{cases}$$

The differential $\delta : C_p(V) \to C_{p-1}(V)$ is given by the formula

$$\delta = \sum_{i=0}^{p} (-1)^i \, \partial_i .$$

Lemma

If $X_\bullet$ is a Lie n-groupoid, the vector spaces $C_p(\mathbb{T})$ form a vector bundle $\mathbb{T}^X_p$, and $\mathbb{T}^X_p$ vanishes for $p > n$.

In this way, we obtain a complex $\mathbb{T}^X_*$ of vector bundles, called the **tangent complex** of $X_\bullet$:

$$0 \to \mathbb{T}^X_n \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathbb{T}^X_1 \xrightarrow{\delta} \mathbb{T}^X_0 \cong TX_0 \to 0$$

Example: the tangent complex of a Lie groupoid

When $X_\bullet \cong N_\bullet\mathcal{G}$ is the nerve of a Lie groupoid, this complex may be identified with the anchor map of the Lie algebroid associated to $\mathcal{G}$:

$$A \xrightarrow{a} TG_0$$

In particular, $\mathbb{T}_*^{N_\bullet G}$ may be identified with the graded vector space $\mathfrak{g}[1]$ (the Lie algebra $\mathfrak{g}$, concentrated in degree 1).

The definition of the tangent complex descends to differentiable n-stacks, at least in the derived category, by the following result.

Lemma

A hypercover $f_\bullet : X_\bullet \to Y_\bullet$ of Lie n-groupoids induces a quasi-isomorphism of complexes of vector bundles $\mathbb{T}^X_* \to f_0^* \mathbb{T}^Y_*$.

We now return to the situation of a pre-symplectic n-groupoid. Using the two-form $\Omega_n \in \Omega^2(X_n)$, we may define a graded antisymmetric form on the tangent complex $\mathbb{T}^X_*$, using the shuffle map.

A (p, q)-shuffle $\pi \in Ш(p, q)$ is a permutation of the set $\{0, \dots, p + q - 1\}$ such that $\pi(i) < \pi(j)$ if $i < j < p$ or $p \leqslant i < j$.

In particular, $Ш(p, q)$ has cardinality $\binom{p+q}{p}$. For more details on shuffles, see the papers of Eilenberg and Moore cited above.

Suppose that $v \in T_{(\sigma_0)^p x} X_p$ and $w \in T_{(\sigma_0)^q x} X_q$ are tangent vectors, representing vectors in the cotangent complex at $x \in X_0$ of degrees $p + q = n$.

Define the pairing of v and w as a sum over shuffles:

$$\langle v, w \rangle = \sum_{\pi \in Ш(p,q)} (-1)^\pi \, \Omega_n\big(\big(\sigma_{\pi(p+q-1)} \dots \sigma_{\pi(p)}\big)_* v, \big(\sigma_{\pi(p-1)} \dots \sigma_{\pi(0)}\big)_* w\big)$$

Here, $(-1)^\pi$ is the sign character of the shuffle π, viewed as an element of the permutation group S_{p+q}.

Exercise

Check that this bilinear form is unchanged if Ω is replaced by

$$\tilde{\Omega} = \Omega + D\alpha, \quad \alpha \in F^2 \Omega^{n+1}(X)$$

In order to justify the definition of $\langle v, w\rangle$, we must check three things:

1. this form is graded antisymmetric:

$$\langle w, v\rangle = -(-1)^{pq}\langle v, w\rangle$$

   (this equals $(-1)^{p-1}\langle v, w\rangle$ if n is even, and $-\langle v, w\rangle$ if n is odd);
2. it vanishes on degenerate vectors: if $u \in T_{(\sigma_0)^{p-1}x}X_{p-1}$, then $\langle(\sigma_i)_* u, w\rangle = 0$ for any degeneracy map $\sigma_i : X_{p-1} \to X_p$;
3. $\langle \delta v, w\rangle + (-1)^p\langle v, \delta w\rangle = 0$.

The first of these are true in general; the second requires that Ω_n be normalized; the third uses the multiplicativity of Ω_n.

Definition

A **symplectic** n-**groupoid** is a pre-symplectic n-groupoid $(X_\bullet, \Omega)$ for which the graded antisymmetric form induced by Ω on the homology groups $H_*(\mathbb{T}^X_*)$ is non-degenerate at every point $x \in X_0$.

If $X_\bullet \to Y_\bullet$ is a hypercover of pre-symplectic n-groupoids and either $X_\bullet$ or $Y_\bullet$ is symplectic, then both are.

In the case where $n = 1$, we recover Xu's definition of quasi-symplectic groupoids.

Ping Xu: Momentum maps and Morita equivalence
J. Diff. Geom. **67** (2004), 289–333

In this case, we have a three-form $\Omega_0 \in \Omega^3(G_0)$ and a two-form $\Omega_1 \in \Omega^2(G_1)$, and the two-form defines a map from the Lie algebroid A to the cotangent bundle T^*G_0, which is non-degenerate on the kernel of the anchor map.

The prototypical example of a quasi-symplectic groupoid is the action groupoid associated to the adjoint action of a compact Lie group G on itself. This example is worked out in

A. Alekseev, A. Malkin and E. Meinrenken: Lie group valued moment maps
J. Differential Geom. **48** (1998), 445–495

Many more examples of quasi-symplectic groupoids are constructed in

Alberto Cattaneo and Ping Xu: Integration of twisted Poisson structures
J. Geom. Phys. **49** (2004), 187–196

If Ω_0 vanishes, then Ω_1 is a closed two-form, and we retrieve Weinstein's definition of a symplectic groupoid. Xu's definition improves on the original definition, in the sense that it is invariant under formation of hypercovers, while Weinstein's definition of a symplectic groupoid is not.

Symplectic 2-groupoids

At last, we return to our favourite Lie groupoid, the nerve $N_\bullet\,\mathrm{U}(N)$ of the unitary group. As we saw in the second lecture, this simplicial manifold carries a closed 2-form $\Phi(c_2) \in F^2\Omega^4(N_\bullet\,\mathrm{U}(N))$. The associated 2-form on $N_2\,\mathrm{U}(N)$ is given by the formula

$$\Phi_2(P) = -\frac{1}{8\pi^2}\Big(\mathrm{Tr}\big(((dg_1)g_1^{-1})(g_2^{-1}(dg_2))\big) - \mathrm{Tr}(g_1^{-1}dg_1) \wedge \mathrm{Tr}(g_2^{-1}dg_2)\Big)$$

This symplectic form is non-degenerate if $N > 1$ (and vanishes if $N = 1$).

Exercise

Consider the differential form $\Phi(P) \in F^2\Omega^4(N_\bullet G)$ associated by Shulman to a non-degenerate invariant inner product $P \in I^2(G)$ on the Lie algebra $\mathfrak{g}$ of a compact Lie group G. Prove that $\Phi(P)$ is a symplectic form on $N_\bullet G$, making it into a symplectic 2-groupoid.

The inertia stack

Let S^1 be the simplicial circle, obtained by identifying the endpoints of the 1-simplex Δ^1. Thus, S^1 has $n+1$ n-simplices for all $n \geqslant 0$, and precisely two of them are non-degenerate, one each in dimensions 0 and 1.

Definition

The **inertia** groupoid of an n-group is the n-groupoid $\mathrm{Map}(S^1, X)$, whose set of n-simplices is given by the manifold of simplicial maps

$$\mathrm{Map}(S^1, X)_n = \mathrm{Hom}(S^1 \times \Delta^n, X).$$

This definition is quite subtle, because the simplicial set $S^1 \times \Delta^n$ is rather complicated: it is obtained by gluing $n+1$ copies of the $(n+1)$-simplex Δ^{n+1}.

The definition of the inertia stack does not extend to Lie n-groupoids: the problem is that, in general, $\mathrm{Hom}(S^1 \times \Delta^n, X)$ is not a manifold.

Toën and Vezossi have shown that the inertia stack of a Lie n-groupoid is a derived stack (roughly speaking, a differential graded Lie ∞-groupoid). More generally, they show that for any finite simplicial set $K_\bullet$, the simplicial mapping space $\mathrm{Map}(K, X)$ is a derived stack.

The tangent complex of a derived stack is a bounded complex extending in both the positive and negative directions. There is a quasi-isomorphism

$$\mathbb{T}_*^{\mathrm{Map}(K,X)} \simeq C^*(K) \otimes \mathbb{T}_*^X .$$

In particular,

$$H_p(\mathbb{T}_*^{\mathrm{Map}(S^1,X)}) \cong H_p(\mathbb{T}_*^X) \cong H_{p+1}(\mathbb{T}_*^X)$$

Pantev et al. show by a simplicial analogue of the AKSZ construction of topological field theories that if $X_\bullet$ is a symplectic n-groupoid and $K_\bullet$ is a triangulation of an oriented closed m-manifold, there is a symplectic form of degree $n-m+2$ on the derived stack $\mathrm{Map}(K,X)$.

Roughly speaking, the simplicial AKSZ construction bears the same relation to the original AKSZ construction as simplicial cochains bear to differential forms.

In the special case where $X_\bullet$ is the symplectic 2-groupoid $N_\bullet G$, the inertia stack may be identified with the action groupoid for the adjoint action of G. (In this case, the tangent complex is easily seen to vanish in negative degrees.) The symplectic form on this Lie groupoid obtained via the simplicial AKSZ construction using the Shulman's symplectic form on $N_\bullet G$ may be identifed with the symplectic form constructed by Alekseev, Malkin and Meinrenken.