NOTES ON DIFFERENTIAL COHOMOLOGY AND GERBES

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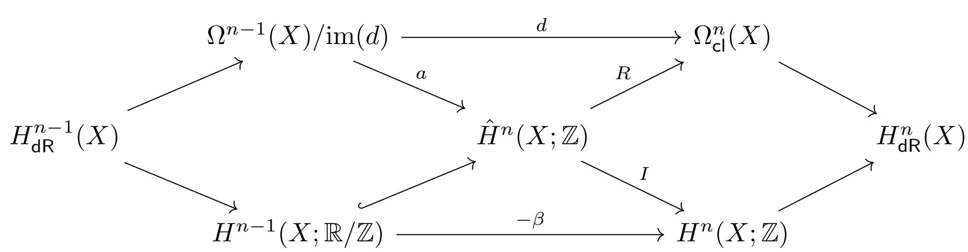
Contents

1. Deligne cohomology

1.1. **Differential cohomology.** Ordinary differential cohomology is a refinement of ordinary (integral) cohomology for manifolds in the sense that it retains *geometric* data in the guise of differential forms. Differential cohomology theories do not fall under the umbrella of (generalized) cohomology theories as they follow fundamentally different axioms. We will focus here on ordinary differential cohomology and in particular a rather workable model known as smooth Deligne cohomology. Before we get into details, however, it is worth keeping in mind some generalities.

Ordinary differential cohomology fits into a diagram



where the diagonal sequences are exact and the top and bottom are long exact (the bottom being part of the Bockstein sequence). This diagram shows that ordinary differential cohomology is fundamentally determined by the interplay between differential forms, de Rham cohomology, and singular cohomology and in fact the axiomatics can be stated in terms of this diagram.

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More generally, Bunke Nikolaus and Volkl have shown that any sheaf on the site of manifolds valued in a stable infinity-category gives rise to such a diagram/theory. In this sense differential cohomology is the natural cohomology theory for manifolds.

1.2. **The Deligne model.** Let's explain Deligne's model for differential cohomology. The k**th Deligne complex** (for $k \geq 1$) is the cochain complex of sheaves

$$\mathbb{Z}_{D,\infty}(k) = \underline{\mathbb{Z}} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \to \cdots \to \Omega^{k-1}.$$

The k**th Deligne cohomology** is the kth sheaf (hyper)cohomology group of the kth Deligne complex:

$$\hat{H}^k(X;\mathbb{Z}) := H^k(X;\mathbb{Z}_{D,\infty}(k)^\bullet).$$

This is a rather abstract definition, but we can always compute these groups by writing a Čech resolution of the complex, say, after choosing a fine enough open cover of our manifold X.[1] Without having to do that, however, it is clear that

$$\hat{H}^0(X;\mathbb{Z}) = H^0(X;\underline{\mathbb{Z}}[0]) = H^0(X;\mathbb{Z}).$$

Let's compute the next simplest example. We will be dropping the underline under the constant sheaf $\underline{\mathbb{Z}}$ for ease of notation. Consider $\mathbb{Z}_{D,\infty}(1) = \underline{\mathbb{Z}} \to \Omega^0$. We obtain a double complex coming from (applying global sections to) the Čech resolution:

$$\begin{array}{ccc}
\check{C}^0(\mathbb{Z}) & \longrightarrow & \check{C}^0(\Omega^0) \\
\downarrow & & \downarrow \\
\check{C}^1(\mathbb{Z}) & \longrightarrow & \check{C}^1(\Omega^0) \\
\downarrow & & \downarrow \\
\check{C}^2(\mathbb{Z}) & \longrightarrow & \check{C}^2(\Omega^0) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

The first sheaf cohomology can now be written as the first cochain cohomology of the associated total complex. If we denote the horizontal differential by d and the vertical differential by δ, the differential in the total complex is given

$$D = d + (-1)^p \delta$$

where p is the horizontal degree. In particular we compute the cohomology at

$$\check{C}^0(\mathbb{Z}) \to \check{C}^0(\Omega^0) \oplus \check{C}^1(\mathbb{Z}) \to \check{C}^1(\Omega^0) \oplus \check{C}^2(\mathbb{Z})$$

Notice that the second map sends

$$(f_\alpha, n_{\alpha\beta}) \mapsto (f_\alpha - f_\beta + n_{\alpha\beta}, n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha\beta}).$$

Hence the kernel consists of the data of real-valued functions on U_α that glue up to an integer, with the integer satisfying the expected cocycle condition. This data patches together to yield a smooth map $X \to U(1)$. On the other hand, the image of the first map is the data of a locally constant integral-valued function on each U_α together with, on overlaps, the data of the difference in these integers. According to

[1] Recall that by Riemannian geometry we can always find a differentiably good open cover of X, i.e. one where the opens and all intersections are contractible.

the interpretation as a map to $U(1)$ these yield the trivial map $X \to U(1)$ sending $x \mapsto 1$ for all $x \in X$. We conclude that

$$H^1(X; \mathbb{Z}_{D,\infty}(1)) \cong C^\infty(X, U(1)).$$

Let's next try to recover line bundles with connection. We have the complex of sheaves

$$\mathbb{Z}_{D,\infty}(2) = \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1.$$

Again using a Čech resolution, we have

$$\begin{array}{ccccc}
\check{C}^0(\mathbb{Z}) & \longrightarrow & \check{C}^0(\Omega^0) & \longrightarrow & \check{C}^0(\Omega^1) \\
\downarrow & & \downarrow & & \downarrow \\
\check{C}^1(\mathbb{Z}) & \longrightarrow & \check{C}^1(\Omega^0) & \longrightarrow & \check{C}^1(\Omega^1) \\
\downarrow & & \downarrow & & \downarrow \\
\check{C}^2(\mathbb{Z}) & \longrightarrow & \check{C}^2(\Omega^0) & \longrightarrow & \check{C}^2(\Omega^1) \\
\downarrow & & \downarrow & & \downarrow \\
\check{C}^3(\mathbb{Z}) & \longrightarrow & \check{C}^3(\Omega^0) & \longrightarrow & \check{C}^3(\Omega^1) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

We are interested in the cohomology of the total complex at:

$$\check{C}^0(\Omega^0) \oplus \check{C}^1(\mathbb{Z}) \to \check{C}^0(\Omega^1) \oplus \check{C}^1(\Omega^0) \oplus \check{C}^2(\mathbb{Z}) \to \check{C}^1(\Omega^1) \oplus \check{C}^2(\Omega^0) \oplus \check{C}^3(\mathbb{Z}).$$

The second map is given

$$(A_\alpha, f_{\alpha\beta}, n_{\alpha\beta\gamma}) \mapsto (A_\beta - A_\alpha + df_{\alpha\beta}, -f_{\beta\gamma} + f_{\alpha\gamma} - f_{\alpha\beta} + n_{\alpha\beta\gamma}, \delta n).$$

Let's look first at the second component. The requirement that the second component vanish allows us to construct a line bundle $L \to X$ with transition functions given $\exp(2\pi i f_{\alpha\beta}) : U_\alpha \cap U_\beta \to U(1)$. We now use the one-forms A_α to construct a connection on this line bundle. Fix trivializing sections s_α on U_α. By construction we have that

$$s_\alpha = e^{2\pi i f_{\alpha\beta}} s_\beta.$$

Define, on U_α the connection $\nabla_\alpha = d + 2\pi i A_\alpha$. These connections glue to a connection on L: Fix the sign.

$$\begin{aligned}
\nabla_\alpha s_\alpha = (d + 2\pi i A_\alpha)(e^{2\pi i f_{\alpha\beta}} s_\beta) &= 2\pi i df_{\alpha\beta} e^{2\pi i f_{\alpha\beta}} s_\beta + 2\pi i A_\alpha s_\beta \\
&= 2\pi i A_\beta e^{2\pi i f_{\alpha\beta}} s_\beta = e^{2\pi i f_{\alpha\beta}} \nabla_\beta s_\beta,
\end{aligned}$$

as desired. Here we have used that $A_\beta - A_\alpha + df_{\alpha\beta} = 0$. To conclude that degree two Deligne cohomology computes isomorphism classes of complex line bundles with connection it remains to check that the image of the first map yields trivial bundles equipped with trivial connection. The map sends

$$(f_\alpha, n_{\alpha\beta}) \mapsto (df_\alpha, f_\alpha - f_\beta + n_{\alpha\beta}, n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha,\beta}).$$

The transition functions induced by this data are of the form $\exp(2\pi i(f_\alpha - f_\beta))$. Choosing trivializations s_α on U_α we obtain a global trivialization given on U_α by $e^{2\pi i f_\alpha} s_\alpha$ because

There's a sign issue here...

$$e^{2\pi i f_\alpha} s_\alpha = e^{2\pi i f_\alpha} e^{2\pi i(f_\beta - f_\alpha)} s_\beta = e^{2\pi i f_\beta} s_\beta.$$

Moreover the connection defined as above, $\nabla_\alpha = d + 2\pi i df_\alpha$, is trivial on these sections:

Another sign wrong...

$$\nabla_\alpha(e^{2\pi i f_\alpha} s_\alpha) = 0.$$

Thus we conclude that

What's up with the hermitian metric?

$$H^2(X; \mathbb{Z}_{D,\infty}(2)) \cong \{\text{line bundles with connection}\}/\sim.$$

What should the third Deligne cohomology group compute? Well $U(1)$ functions are glued together up to integers while line bundles are glued together up to $U(1)$ functions. It is natural to expect, then, that the degree three objects should be glued together using line bundles. This will lead us to the notion of a bundle gerbe.

1.3. **Ring structure.** Just as ordinary cohomology has a natural ring structure, it turns out that ordinary differential cohomology is also equipped with a multiplication. Recalling our computations from above this means that the product should give us a way of constructing a line bundle from two $U(1)$-functions, a gerbe from a line bundle and a $U(1)$-function, etc. We can obtain this multiplication from a rather straightforward (if somewhat mysterious) multiplicative structure on the Deligne complexes, known as the Beilinson-Deligne cup product:

$$\smile : \mathbb{Z}_{D,\infty}(k) \otimes \mathbb{Z}_{D,\infty}(\ell) \to \mathbb{Z}_{D,\infty}(k+\ell)$$

given by (over each open set)

$$x \smile y = \begin{cases} x \cdot y & \deg x = 0 \\ x \wedge dy & \deg x > 0, \deg y = \ell \\ 0 & \text{otherwise} \end{cases}$$

There is the slight complication that the tensor product of sheaves need not be a sheaf, but we can define the above map as a map of presheaves and since the target is a sheaf we obtain an induce a map of sheaves by the universal property of sheafification which will satisfy all the properties of the map of presheaves.

The following properties are not very difficult to check but we write them out to get some concrete practice working with the product.

Proposition 1. *The BD cup product enjoys the following properties:*

(a) it is a map of complexes, i.e. it satisfies the (graded) Leibniz rule,
(b) it is associative,
(c) it is (graded) commutative up to homotopy,
(d) 1 is a left unit and hence a right unit up to homotopy.

Proof. We will use $\tilde{d}$ to denote the differential on the Deligne complex to distinguish it from the de Rham differential (they differ of course only in degree 0). The graded Leibniz rule requires that

$$\tilde{d}(x \smile y) = \smile(\tilde{d}(x \otimes y)) = \tilde{d}x \smile y + (-1)^{|x|} x \smile \tilde{d}y.$$

There are three cases: the first is where $|x| = 0$, the second is where $|x| \neq 0$ and $|y| = \ell$, and the third is where $|x| \neq 0$ and $|y| \neq \ell$. In the first case both sides of the above equation equal $x\tilde{d}y$ and in the second case both equal $dx \wedge dy$. For the third case if $|y| \neq \ell - 1$ then we have zero on both sides. If $|y| = \ell - 1$ then the left is zero and the right is $(-1)^{|x|} x \wedge d^2 y = 0$ as well.

For associativity one checks that the diagram

$$\begin{array}{ccc} \mathbb{Z}_{D,\infty}(k) \otimes \mathbb{Z}_{D,\infty}(\ell) \otimes \mathbb{Z}_{D,\infty}(m) & \xrightarrow{\mathrm{id} \otimes \smile} & \mathbb{Z}_{D,\infty}(k) \otimes \mathbb{Z}_{D,\infty}(\ell + m) \\ \big\downarrow {\scriptstyle \smile \otimes \mathrm{id}} & & \big\downarrow {\scriptstyle \smile} \\ \mathbb{Z}_{D,\infty}(k+\ell) \otimes \mathbb{Z}_{D,\infty}(m) & \xrightarrow{\quad \smile \quad} & \mathbb{Z}_{D,\infty}(k+\ell+m) \end{array}$$

commutes. Finish this.

Finally we check that the diagram

$$\begin{array}{ccc} \mathbb{Z}_{D,\infty}(k) \otimes \mathbb{Z}_{D,\infty}(\ell) & \xrightarrow{\smile} & \mathbb{Z}_{D,\infty}(k+\ell) \\ \big\downarrow {\scriptstyle \tau} & \nearrow_{\smile} & \\ \mathbb{Z}_{D,\infty}(\ell) \otimes \mathbb{Z}_{D,\infty}(k) & & \end{array}$$

commutes up to homotopy where $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$, i.e. there exists a cochain homotopy h such that

$$x \smile y - (-1)^{|x||y|} y \smile x = dh + hd$$

as maps $\mathbb{Z}_{D,\infty}(k) \otimes \mathbb{Z}_{D,\infty}(\ell) \to \mathbb{Z}_{D,\infty}(k+\ell)$. Our candidate for h is the following:

$$h(x \otimes y) = \begin{cases} 0 & |x| = 0 \text{ or } |y| = 0 \\ -(-1)^{|x|} x \wedge y & \text{otherwise.} \end{cases}$$

For the sake of concreteness let us check explicitly that h does indeed provide a cochain homotopy. There are nine cases to check (three cases for each case of x and y as given in the definition of the multiplication). Let us write $\iota : \mathbb{Z} \hookrightarrow \Omega^0$ for the inclusion of integers (we may assume our open is connected) into the smooth functions. One has to be careful to note that $|\cdot|$ denotes degree in the Deligne complex and not the differential form degree. Moreover one should distinguish between the Deligne differential in the source and target (of h). The cases are:

(1) $|x| = 0$ and $|y| = 0$. Then the left is $xy - yx = 0$ and the right is $h(\iota(x) \otimes y) + h(x \otimes \iota(y)) = \iota(x)y - x\iota(y) = 0$.
(2) $|x| = 0$ and $0 < |y| < \ell$. Then the left is $xy - 0 = xy$ and the right is $h(\iota(x) \otimes y) = \iota(x)y$.
(3) $|x| = 0$ and $|y| = \ell$. Then the left is $xy - 0 = xy$ and the right is $h(\iota(x) \otimes y) = \iota(x)y$.
(4) $0 < |x| < k$ and $|y| = 0$. Then the left is $0 - yx = -yx$ and the right is $(-1)^{|x|} h(x \otimes \iota(y)) = -x\iota(y) = -\iota(y)x$.
(5) $0 < |x| < k$ and $0 < |y| < \ell$. Then the left is 0 and the right is $-(-1)^{|x|} \tilde{d}x \wedge y - (-1)^{|x|}(-1)^{|x|-1} x \wedge \tilde{d}y - (-1)^{|x|+1} \tilde{d}x \wedge y - (-1)^{|x|}(-1)^{|x|} x \wedge \tilde{d}y = 0$.
(6) $0 < |x| < k$ and $|y| = \ell$. Then the left is $x \wedge dy - 0$ and the right is $-(-1)^{|x|} \tilde{d}x \wedge y - (-1)^{|x|}(-1)^{|x|-1} x \wedge \tilde{d}y - (-1)^{|x|+1} dx \wedge y + 0 = x \wedge dy$.
(7) $|x| = k$ and $|y| = 0$. Then the left is $-yx$ and the right is $-x\iota(y)$.
(8) $|x| = k$ and $0 < |y| < \ell$. Then the left is $-(-1)^{|x||y|} y \wedge dx$ and the right is $-(-1)^{|x||y|} y \wedge dx$.

(9) $|x| = k$ and $|y| = \ell$. Then the left is $x \wedge dy - (-1)^{|x||y|} y \wedge dx$ and the right is $-(-1)^{|x|}\tilde{dx} \wedge y - (-1)^{|x|}(-1)^{|x|-1} x \wedge \tilde{dy}$, which simplifies to the left.

We conclude that the BD cup product is commutative up to homotopy. The last statement is clear. □

Remark 2. Notice that while the homotopy h constructed in the proof above is a binary operation, it is not associative. For example, if we take $x \otimes y \otimes z$ with x, y, z homogeneous of nonzero degree then

$$h(h(x \otimes y) \otimes z) = -(-1)^{|x|} h(x \wedge y \otimes z) = (-1)^{|x|}(-1)^{|x|+|y|-1} x \wedge y \wedge z$$

whereas

$$h(x \otimes h(y \otimes z)) = -(-1)^{|y|} h(x \otimes y \wedge z) = (-1)^{|x|+|y|} x \wedge y \wedge z,$$

which differs by a sign $-(-1)^{|x|}$.

Moreover h is not (graded) commutative. By definition, for x, y homogeneous of nonzero degree

$$h(x \otimes y) = -(-1)^{|x|} x \wedge y$$

whereas

$$\begin{aligned}(-1)^{|x||y|} h(y \otimes x) &= -(-1)^{|x||y|}(-1)^{|y|} y \wedge x \\ &= -(-1)^{|y|(|x|+1)}(-1)^{(|x|+1)(|y|+1)} x \wedge y \\ &= (-1)^{|x|} x \wedge y.\end{aligned}$$

In particular we see that h is anti-graded-commutative.

A similar situation arises when defining the cup product on singular cochains (or similar the cup product on Čech cohomology). There the product is commutative up to a homotopy known as the cup-1 product $\smile_1$, due to Steenrod. This product is in turn commutative up to a homotopy known as the cup-2 product, and so on. In other words, the cup product is commutative up to a family of coherent homotopies. This yields, in particular, the structure of an $\mathcal{E}_\infty$-algebra on singular cochains (for explicit construction of the operad via these higher cup products see the paper of McClure and Smith). More abstractly, the $\mathcal{E}_\infty$-structure arises from the fact that every space is a cocommutative comonoid via the diagonal map, as explained to me by Elden: the singular cochains functor from $\mathsf{Spaces}^{\mathsf{op}} \to \mathsf{Ch}$ is monoidal up to homotopy (this is the Eilenberg-Zilber theorem) and it is a fact that such functors preserve commutative (up to homotopy) monoid objects. This last fact is some sort of homotopy transfer result, e.g. commutative algebra structures can be transferred over weak equivalences to yield $\mathcal{E}_\infty$-structures.

Do we have a similar structure on the Deligne complex? To do this concretely would require us to write down higher BD cup products. We have written down the analog of the $\smile_1$ product as h above, but I'm not sure how to write down the higher homotopies. Alternatively we might look for an abstract origin for an $\mathcal{E}_\infty$ structure whose zeroth order product is the BD cup product. Beilinson, in the paper where he introduces the BD product, remarks that the product is precisely coming from the fact that the Deligne complex can be written as a homotopy pullback of commutative dgas $\mathbb{Z} \to \Omega^\bullet \leftarrow \Omega^{\geq k}$ and that this induces an "Alexander-Whitney" product on the Deligne complex. Again, this is some sort of homotopy transfer result.

Abstractions aside, it is instructive to write down explicit Čech formulas for the product of Deligne cocycles. Recall that we can compute kth Deligne cohomology as the kth cohomology of the total complex of a double complex which is a Čech resolution of the kth Deligne complex. In particular we are interested in the cohomology at

$$\cdots \to \bigoplus_{i+j=k-1} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(k)^j) \to \bigoplus_{i+j=k} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(k)^j) \to \bigoplus_{i+j=k+1} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(k)^j) \to \cdots$$

where the differentials are the usual combination of the de Rham and Čech differentials. We obtain a map

$$\left(\bigoplus_{i+j=k} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(k)^j)\right) \otimes \left(\bigoplus_{i+j=\ell} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(\ell)^j)\right) \to \bigoplus_{i+j+k+\ell} \check{C}^i(\mathcal{U};\mathbb{Z}_{D,\infty}(k+\ell)^j)$$

using (components of) the cup product defined above together with the Čech cup product. One has to check that the product of cocycles is again a cocycle and that multiplying against coboundaries yields coboundaries.

For simplicity and concreteness let us just check this for the case where we multiply two degree-one cocycles. Geometrically this will correspond to constructing a line bundle (up to isomorphism) from two smooth $U(1)$-valued functions. Recall that

$$\mathbb{Z}_{D,\infty}(1) = \mathbb{Z} \hookrightarrow \Omega^0, \qquad \mathbb{Z}_{D,\infty}(2) = \mathbb{Z} \hookrightarrow \Omega^0 \to \Omega^1.$$

Thus we wish to write down explicitly the multiplication map

$$\left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right)^{\otimes 2} \xrightarrow{\smile} \check{C}^0(\mathcal{U};\Omega^1) \oplus \check{C}^1(\mathcal{U};\Omega^0) \oplus \check{C}^2(\mathcal{U};\mathbb{Z})$$

induced by the cup product $\smile$ defined above. Consider an element $(f,n) \otimes (g,m)$ on the left. Expanding the tensor product we first have a map

$$\check{C}^0(\mathcal{U};\Omega^0) \otimes \check{C}^0(\mathcal{U};\Omega^0) \to \check{C}^0(\Omega^1)$$

which sends $f \otimes g$ to the Čech zero-cochain

$$(f \smile g)_\alpha = f_\alpha \smile g_\alpha = f_\alpha dg_\alpha.$$

by the definition of the product above. Next we have a map

$$\check{C}^0(\mathcal{U};\Omega^0) \otimes \check{C}^1(\mathcal{U};\mathbb{Z}) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}) \otimes \check{C}^0(\mathcal{U};\Omega^0) \to \check{C}^1(\mathcal{U};\Omega^0)$$

sending $n \otimes g + f \otimes m$ to the Čech one-cochain

$$(n \smile g)_{\alpha\beta} = n_{\alpha\beta} g_\beta$$
$$(f \smile m)_{\alpha\beta} = 0.$$

Finally we have a map

$$\check{C}^1(\mathcal{U};\mathbb{Z}) \otimes \check{C}^1(\mathcal{U};\mathbb{Z}) \to \check{C}^2(\mathcal{U};\mathbb{Z})$$

given by sending $n \otimes m$ to the Čech two-cochain

$$(n \smile m)_{\alpha\beta\gamma} = n_{\alpha\beta} m_{\beta\gamma}.$$

These are the explicit formulas for the product of two Deligne one-cochains. Let us check that the product of cocycles is a cocycle. Recall that for (f,n) and (g,m) to be cocycles we require that

$$f_\alpha - f_\beta + n_{\alpha\beta} = 0, \qquad n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha\beta} = 0$$
$$g_\alpha - g_\beta + m_{\alpha\beta} = 0, \qquad m_{\beta\gamma} - m_{\alpha\gamma} + m_{\alpha\beta} = 0.$$

We wish to check that applying the total complex differential to $f \smile g + n \smile g + n \smile m$ yields zero. Using the formulas for this differential derived above (when computed degree two Deligne cohomology) we find that we must have

$$0 = f_\beta dg_\beta - f_\alpha dg_\alpha + n_{\alpha\beta} dg_\beta$$
$$0 = -n_{\beta\gamma} g_\gamma + n_{\alpha\gamma} g_\gamma - n_{\alpha\beta} g_\beta + n_{\alpha\beta} m_{\beta\gamma}$$
$$0 = n_{\beta\gamma} m_{\gamma\delta} - n_{\alpha\gamma} m_{\gamma\delta} + n_{\alpha\beta} m_{\beta\delta} - n_{\alpha\beta} m_{\beta\gamma}.$$

There are some sign errors! Perhaps in the earlier formulas.

It is easy to check that these follow from the cocycle conditions for (f, n) and (g, m) above. Finally let us check that the product of a coboundary and a cocycle is a coboundary. If (f, n) is a coboundary this means there exists a Čech zero-cycle with coefficients in the constant sheaf $\mathbb{Z}$, denote it by a, such that

$$f_\alpha = a_\alpha \qquad n_{\alpha\beta} = a_\beta - a_\alpha.$$

We thus obtain, using that (g, m) is a cocycle,

$$(f \smile g)_\alpha = f_\alpha dg_\alpha = a_\alpha dg_\alpha$$
$$(n \smile g)_{\alpha\beta} = a_\beta g_\beta - a_\alpha g_\beta = a_\beta g_\beta - a_\alpha g_\alpha - a_\alpha m_{\alpha\beta}$$
$$(n \smile m)_{\alpha\beta\gamma} = n_{\alpha\beta} m_{\beta\gamma} = a_\beta m_{\beta\gamma} - a_\alpha m_{\beta\gamma}.$$

One checks (again using the formulas from the computation of degree two Deligne cohomology) that this defines a coboundary which is the total differential of $(-a_\alpha g_\alpha, -a_\alpha m_{\alpha\beta})$.

Again, up to some pesky signs.

Recalling the interpretation of degree one and two Deligne cohomologies as smooth $U(1)$-valued functions and line bundles with connection up to isomorphism, we see that the product yields, for any two smooth $U(1)$-valued functions, a line bundle with connection. This line bundle can be described explicitly using the Čech description as above: the transition functions are given data (f, n) and (g, m) defining $U(1)$-functions, the line bundle has transition functions $n_{\alpha\beta} g_\beta : U_\alpha \cap U_\beta \to U(1)$ and connection one-forms $2\pi i f_\alpha dg_\alpha$ on U_α.

Go over this carefully, something looks funny

We see that the product gives us a systematic procedure for constructing higher degree classes in Deligne cohomology. The geometric interpretation of these classes is somewhat unclear, though. As we will see later, the objects that are classified by higher degree classes behave higher-categorically, which become quite difficult to understand concretely. For instance, $U(1)$-functions form a set, line bundles form a 1-category, gerbes form a 2-category, etc.

1.4. **Fiber integration.**

1.5. **Application: families of Dirac operators.**

Differential cohomology arises naturally in index theory in various forms. We will discuss degree one and degree two classes that naturally arise when considering families of Dirac operators.

The geometric setup is the following. Let $\pi : M \to B$ be a smooth family of spin manifolds of dimension n. Denote by $T(M/B) \to M$ the vertical tangent bundle of π, i.e. the bundle of vectors in the kernel of the pushforward π_*. The spin structure on the fibers M_b is the data of a spin structure on $T(M/B)$. The usual representation theory of Clifford algebras yields a spinor bundle $S(M/B) \to M$

associated to the Clifford bundle $C(T(M/B))$. If $n = 2k$, i.e. we have an even-dimensional family of manifolds, then the spinor bundle is naturally $\mathbb{Z}/2$-graded. In the odd-dimensional case, on the other hand, the spinor bundle is ungraded, and we manually define $S(M/B)$ to be the $\mathbb{Z}/2$-graded bundle given as a direct sum of two copies of the ungraded spinor bundle. Either way we obtain a $\mathbb{Z}/2$-graded vector bundle that we will denote by $S(M/B)$. The Clifford multiplication and the Levi-Civita connection on $T(M/B)$ yield a first-order odd differential operator

$$\mathsf{D}^{\pm} : S^{\pm}(M/B) \to S^{\mp}(M/B)$$

that squares to a generalized Laplacian. Here again in the odd case we have no grading so we take the ungraded Dirac operator and view it as a map between the two copies of the spinor bundle. We can twist all the constructions in this section by an auxillary vector bundle with metric and connection on M, though we will refrain from doing so here.

For n odd there is a natural degree one Deligne cocycle on the base B that patches together to yield a $U(1)$ function on B. We construct it as follows. Denote by ω the volume form along the fibers of π and normalize it such that $c(\omega)^2 = 1$. Notice that Clifford multiplication by ω commutes with the Dirac operator and is an odd operator whence we obtain an even operator

$$c(\omega)\mathsf{D} : S^{\pm}(M/B) \to S^{\pm}(M/B)$$

such that $(c(\omega)\mathsf{D})^2 = \mathsf{D}^2$. As such, this operator has a discrete spectrum unbounded positively and negatively.

We define an open cover $\{V_\alpha\}_{\alpha\in\mathbb{R}}$ of B by

$$V_\alpha = \{b \in B \mid \alpha \notin \mathsf{spec}^+(c(\omega)_b\mathsf{D}_b)\}.$$

i.e. V_α is the subset of $b \in B$ over which $\alpha \in \mathbb{R}$ is not an eigenvalue of $c(\omega)_b\mathsf{D}_b$. Straightforward functional analytic arguments show that this does indeed give us an open cover. We define $\eta_\alpha : V_\alpha \to \mathbb{R}$ to be the zeta-function regularization of the difference of the number of eigenvalues (of $c(\omega)_b\mathsf{D}_b$) below and above α. It is a nontrivial result (say using heat kernel or pseudodifferential techniques) to show that this does in fact give us a smooth real-valued function. That n is odd is crucial here—for instance in the heat kernel approach I believe it is needed to guarantee the regularity of the zeta-function regularization at $z = 0$. Moreover for $\alpha < \beta$ we have the relation

$$\frac{1}{2}\eta_\beta(b) = \frac{1}{2}\eta_\alpha(b) - |\{\lambda \in \mathsf{spec}^+(c(\omega)_b\mathsf{D}_b) \mid \alpha < \lambda < \beta\}|$$

over $V_\alpha \cap V_\beta$. The second term on the right is of course an integer $n_{\alpha\beta}$, which yields our Čech-de Rham 1-cocycle $(\eta_\alpha, n_{\alpha\beta})$, which glues to yield a function $\tau_{\mathsf{D}} : B \to U(1)$ that on V_α is given

$$\tau_{\mathsf{D}}(b) = \exp(\pi i \eta_\alpha(b)).$$

In the even-dimensional case one there is a natural degree two Čech-de Rham cocycle known as the determinant line bundle $L \to B$.[2] The construction of the Hermitian line bundle for families of Riemann surfaces is due to Quillen, and in full generality to Bismut and Freed, who additionally constructed a unitary connection. Heuristically the determinant line bundle is given as $(\det\ker\mathsf{D}^+)^\vee \otimes \det\ker\mathsf{D}^-$. Of

[2]In fact this cocycle can be written down in the odd-dimensional case as well, though the resulting geometric line bundle is relatively trivial, c.f. Freed/Moore.

course, $\ker \mathsf{D}^{\pm}$ need not be smooth vector bundles (the kernel may jump rank), so this is at best an intuitive picture.

Define the infinite-rank superbundle $\mathcal{H} = \pi_* S(M/B) \to B$ as having fiber over b the space $\Gamma(M_b, S(M/B))$. Then D becomes an odd endomorphism of $\mathcal{H}$. We define an open cover $\{U_\alpha\}_{\alpha \geq 0}$ of B by

$$U_\alpha = \{b \in B \mid \alpha \notin \mathsf{spec}^+(\mathsf{D}_b^- \mathsf{D}_b^+)\}$$

and superbundles $\mathcal{H}[0, \alpha) \to U_\alpha$ consisting of the sum of the eigenspaces of D^2 for eigenvalue less than α. Functional analytic arguments show that this is indeed a finite-rank vector bundle. Suppose now that we are given a connection on the family $\pi : M \to B$. Then these superbundles naturally obtain metrics and connections (from the spin connection and the connection on the family). Define over U_α,

$$\det \mathcal{H}[0, \alpha) := (\det \mathcal{H}^+[0, \alpha))^\vee \otimes \det \mathcal{H}^-[0, \alpha)$$

which is again naturally equipped with a metric and connection. Over the overlap $U_\alpha \cap U_\beta$ for $\alpha < \beta$ one finds that

$$\det \mathcal{H}[0, \beta) \cong \det \mathcal{H}[0, \alpha) \otimes \det \mathcal{H}(\alpha, \beta).$$

The bundle $\det \mathcal{H}(\alpha, \beta) \to U_\alpha \cap U_\beta$ is trivial as it has a nonvanishing global section $\det \mathsf{D}(\alpha, \beta)$ (since D is of course invertible away from its kernel). Thus these line bundles glue to the determinant line bundle $L \to B$ along these isomorphisms. It is worth remarking that when D has index zero there is a natural global section $\det \mathsf{D} \in \Gamma(B, L)$ that can be interpreted as the determinant of the Dirac operator D. This is important in the Lagrangian formulation of anomalies for quantum field theories with both bosons and fermions.

Unfortunately the metrics and connections on the bundles $\det \mathcal{H}[0, \alpha)$ do not glue to a metric and connection on L. There is some twisting required by zeta-regularized functions—the construction of the metric is due to Quillen and the connection to Bismut and Freed. In fact there is a small miracle that the curvature of this connection is the two-form component of the characteristic class appearing in the families index theorem of Atiyah and Singer. This raises the question as to whether the higher degree components have index-theoretic geometric interpretations. In degree 3 at least there is a construction of a gerbe due to Lott that we will outline below. I believe that the higher degree Deligne cohomology classes are constructed in Bunke's book, though I haven't looked in detail.

Example 3 (Sigma model)**.** Let Σ be a compact surface with spin structure and let X be a Riemannian manifold. Consider a smooth map $B \to \mathrm{Maps}(\Sigma, X)$, which is formally the data of a smooth map

$$M := \Sigma \times B \xrightarrow{\mathrm{ev}} X.$$

Notice that we work over B since $\mathrm{Maps}(\Sigma, X)$ is infinite-dimensional. We have a diagram

$$\begin{array}{ccc} M = \Sigma \times B & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle \pi} & & \\ B & & \end{array}$$

Consider the twist of the vertical spinor bundle on M by $\mathrm{ev}^* TX$. Bunke has shown that if X has a string structure then the Deligne cocycle for the determinant line bundle L can be expressed as the transgression of the first differential fractional Pontryagin class of TX:[3]

$$\hat{c}_1(L) = -\int_{\Sigma\times B/B} \mathrm{ev}^* \left(\frac{\hat{p}_1}{2}(TX)\right)$$

Moreover there is a canonical flat unit-norm trivialization of L determined by the string structure. This is an example of an anomaly cancellation result as well as the power of the formalism of differential cohomology.

2. Gerbes

The notion of a gerbe has been around for quite a while now and there are various approaches to defining them in various contexts. We will discuss a very concrete geometric model due to Murray.

2.1. **Bundle gerbes.**

Just as a line bundle with connection on X is given by a map $L \to X$ of a certain form together with additional data, a bundle gerbe is defined as follows.

Definition 4. A bundle gerbe $\mathcal{G}$ with connective structure on X is the following data:

1. a surjective submersion $U \to X$;
2. a line bundle $L \to U^{[2]}$, where $U^{[k]}$ is the kth fibered product of U with itself over X;
3. a connection ∇^L on L with curvature $F \in \Omega^2(U^{[2]})$;
4. an isomorphism
$$\mu : \pi_{01}^* L \otimes \pi_{12}^* L \xrightarrow{\sim} \pi_{02}^* L$$
of line bundles with connection over $U^{[3]}$ satisfying an "associativity" coherence condition
$$\begin{array}{ccc} \pi_{01}^*L \otimes \pi_{12}^*L \otimes \pi_{23}^*L & \xrightarrow{\pi_{012}^*\mu\otimes \mathrm{id}} & \pi_{02}^*L \otimes \pi_{23}^*L \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{123}^*\mu} & & \Big\downarrow{\scriptstyle \pi_{023}^*\mu} \\ \pi_{01}^*L \otimes \pi_{13}^*L & \xrightarrow{\pi_{013}^*\mu} & \pi_{03}^*L \end{array}$$
over $U^{[4]}$;
5. a two-form $B \in \Omega^2(U)$ called the curving such that $\pi_0^*B - \pi_1^*B = F$.

This is a lot of data to keep track of, so let's look at some examples. The first is the trivial bundle gerbe.

Example 5 (Trivial bundle gerbe)**.** Recall that given a 1-form on X we can construct a topologically trivial line bundle with connection. Now suppose we are instead given a 2-form B on X. Define a bundle gerbe by taking the surjective submersion to be $\mathrm{id} : X \to X$ and the line bundle over $X^{[2]} = X$ to be the trivial line bundle $X \times \mathbb{C} \to X$ with trivial connection. The isomorphism μ is just the identity map between trivial bundles, which of course satisfies the associativity condition. Since $\pi_0 = \pi_1$ the condition on B, $\pi_0^*B - \pi_1^*B = 0$ is trivially satisfied.

[3] Really it's the relative Pfaffian bundle. See Bunke's paper for details.

One might say that a bundle gerbe is trivializable if it is isomorphic to a trivial bundle. However, morphisms of bundle gerbes are somewhat subtle (in fact, bundle gerbes naturally form a 2-category), so we'll refrain from discussing maps for now.

Before we explore more examples, let's sketch why this definition of gerbe matches up with the Deligne description above. Given a sufficiently fine open cover of X and a Deligne 3-cocycle, i.e. the data of $(B_\alpha, A_{\alpha\beta}, f_{\alpha\beta\gamma}, n_{\alpha\beta\gamma\varepsilon})$ in the kernel of the map above, let us construct a bundle gerbe with connective structure. Denote by $\{U_\alpha\}$ the given open cover and let $U = \coprod_\alpha U_\alpha$ with the obvious surjective submersion to X. Define $L \to U^{[2]}$ to be the trivial line bundle with connection $2\pi i A_{\alpha\beta}$ over $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then $-2\pi i B_\alpha \in \Omega^2(U_\alpha)$ is the curving of the gerbe, as the curvature of L over $U_{\alpha\beta}$ is $2\pi i dA_{\alpha\beta}$ and the cocycle condition requires that the difference in curvings over double overlaps is precisely the curvature of L over that overlap.

Next we require an isomorphism of line bundles with connection

$$L_{\alpha\beta} \otimes L_{\alpha\gamma}^{-1} \otimes L_{\beta\gamma} \xrightarrow{\mu} U_{\alpha\beta\gamma} \times \mathbb{C}$$

over $U_{\alpha\beta\gamma}$ for any α, β, γ. We use $f_{\alpha\beta\gamma} \in \Omega^0(U_{\alpha\beta\gamma})$: in particular the triple tensor product on the left is trivial (since L is), whence we can define μ as multiplication by $\exp(2\pi i f_{\alpha\beta\gamma}) : U_{\alpha\beta\gamma} \to U(1)$. It remains to check that μ is compatible with the connections. Take sections $s_{\alpha\beta}, s_{\alpha\gamma}, s_{\beta\gamma}$ of the line bundles on the right and denote by $t_{\alpha\beta\gamma}$ the section of $U_{\alpha\beta\gamma} \times \mathbb{C}$ that is the image under μ:

$$t_{\alpha\beta\gamma} = e^{2\pi i f_{\alpha\beta\gamma}} s_{\alpha\beta} \otimes s_{\alpha\gamma}^{-1} \otimes s_{\beta\gamma}.$$

Then for μ to be compatible with the connections we must have that

$$0 = \nabla(e^{2\pi i f_{\alpha\beta\gamma}} s_{\alpha\beta} \otimes s_{\alpha\gamma}^{-1} \otimes s_{\beta\gamma})$$

since the connection on the trivial bundle is trivial. Differentiating, we find

$$2\pi i e^{2\pi i f_{\alpha\beta\gamma}} df_{\alpha\beta\gamma} s_{\alpha\beta} \otimes s_{\alpha\gamma}^{-1} \otimes s_{\beta\gamma} + 2\pi i e^{2\pi i f_{\alpha\beta\gamma}} (A_{\alpha\beta} - A_{\alpha\gamma} + A_{\beta\gamma}) s_{\alpha\beta} \otimes s_{\alpha\gamma}^{-1} \otimes s_{\beta\gamma} = 0,$$

which is exactly the cocycle condition on $A_{\alpha\beta}$. Finally, the "associativity" of the isomorphism μ on $U^{[4]}$ follows from the cocycle condition on $f_{\alpha\beta\gamma}$.

We omit the verification of the converse: that a bundle gerbe with connective structure determines a Deligne 3-cocycle. We also will refrain for now from discussing isomorphism classes of bundle gerbes.

Remark 6. Let (Y, L, μ) be a bundle gerbe (we'll ignore the connective structure) on X. The bundle gerbe determines a Lie groupoid $\mathcal{G}$ in the following manner. The objects are points $y \in Y$, and the set of morphisms between y_1, y_2 is empty if $\pi(y_1) \neq \pi(y_2)$ in X and the vector space $L_{(y_1,y_2)}$ otherwise. Now let $z_1 : y_1 \to y_2$, $z_2 : y_2 \to y_3$ be two morphisms. The composition $z_2 \circ z_1 : y_1 \to y_3$ is defined by the morphism μ—recall that we have

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \xrightarrow{\sim} \pi_{13}^* L$$

on $Y^{[3]}$ which induces a map $\mu_{(y_1,y_2,y_3)} : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)}$. Thus z_1, z_2 determine a morphism $z_3 \in L_{(y_1,y_3)}$ which we define to be the composition of z_1 and z_2. Composition is associative by the associativity condition on μ (over $Y^{[4]}$). Finally we check that morphisms are invertible: given a map $z : y_1 \to y_2$, maps $w : y_2 \to y_1$ are in bijective correspondence via μ with maps $y_1 \to y_1$. Hence there is a unique map $w : y_2 \to y_1$ such that $w \circ z = \mathrm{id}_{y_1}$ (and similarly for the other composition).

In fact, this allows us to think of a bundle gerbe as the total space of a (bundle associated to) a $BU(1)$-principal 2-bundle. Here $BU(1)$ is thought of as a 2-group. Perhaps we can discuss these notions at a later date.

2.2. **Examples.** We discuss a few examples of bundles gerbes arising naturally from geometry.

Example 7 (Basic gerbe)**.** Let G be a compact, simple, simply-connected Lie group. There is a gerbe on G with curvature 3-form naturally built from the Maurer-Cartan form θ and the (appropriately normalized) invariant bilinear form $\langle -.- \rangle$ on the Lie algebra $\mathfrak{g}$ of G. Recall that the Maurer-Cartan form $\theta \in \Omega^1(G; \mathfrak{g})$ is the left-invariant $\mathfrak{g}$-valued one-form on G defined at the identity T_eG as eating a tangent vector and returning it as an element of the Lie algebra. Define, now,

$$\eta = \lambda_G \langle \theta, [\theta \wedge \theta] \rangle \in \Omega^3(G).$$

Here $[\theta \wedge \theta]$ is a combination of the wedge product of forms and the commutator of Lie algebra elements and $\lambda \in \mathbb{R}$ is a constant chosen such that η defines an integral cohomology class (notice that this constant will depend on our normalization of bilinear form) $[\eta] \in H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ and is the (positive) generator.

The basic gerbe has many different constructions, some of them related to the lifting gerbe we will discuss later. There is a relatively explicit construction for the case of $SU(d+1)$ due to Gawedzki and Ries that was later generalized to G as above by Meinrenken. We will follow the exposition of Waldorf and Schweigert.

To form a surjective submersion (in fact an open cover) over G we use the fact that the fundamental alcove $\mathcal{A}$ in $\mathfrak{g}$ (or equivalently $\mathfrak{g}^\vee$) is in bijection with the conjugacy classes of G. Write $q : G \to \mathcal{A}$ for the quotient map. Writing $\mu_0, \ldots, \mu_d$ for the vertices of $\mathcal{A}$ with $\mu_0 = 0$, define $V_j = q^{-1}(\mathcal{A}_j)$ where $\mathcal{A}_j$ is the open star at μ_j. The V_j yield an open cover of G. To construct line bundles on each V_j, finish

Example 8 (Lott's index gerbe)**.** Given a smooth family of (generalized) Dirac operators D on a family of $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundles over a family of even-dimensional closed manifolds $\mathcal{E} \to M \to B$ one obtains a complex line bundle with Hermitian metric on the parameter space B, denoted

$$\det(\mathsf{D}, \pi_* \mathcal{E}) \to \mathsf{B},$$

known as the determinant line bundle of the family (constructed originally by Quillen). The name comes from the case when $\operatorname{ind} D = 0$, for which there is a section of this line bundle which can be interpreted as the determinant of the family D. There is moreover a connection compatible with the Hermitian metric on the determinant line bundle, due to Bismut and Freed. The curvature of this connection happens to be the degree 2 component of an inhomogeneous form appearing in the transgression formula for the local Atiyah-Singer family index theorem (due to Bismut). It is thus natural to ask whether the higher degree terms also correspond to geometric objects on B.

Lott, in a 2001 paper, given similar data as above in the odd-dimensional case constructs a bundle gerbe with connective structure over B. Whereas the Bismut-Freed connection one-form arises naturally from the determinant, here the relevant quantity will be the eta invariant. I am not very familiar with the relevant odd-dimensional index theory so I will just give a rough outline of the construction. We begin with the geometric setup. Let the fibers of $\pi : M \to B$ be closed odd-dimensional (oriented) manifolds. Suppose we have a spin structure on the vertical tangent bundle and let $\mathcal{E}$ be a complex vector bundle (with metric and compatible connection) over M which is a twist of the spinor bundle by an auxilliary complex vector bundle (with metric and connection). This data yields a smooth family of (generalized) Dirac operators that we will denote $(\mathsf{D}_0)_{\mathsf{b}\in\mathsf{B}}$. Define $\pi_*\mathcal{E}$ to be the infinite-rank vector bundle over B whose fiber at $b \in B$ is the space of smooth sections $\Gamma(M_b, \mathcal{E}|_{M_b})$. If we choose a connection (horizontal distribution) on $M \to B$ we obtain a connection on $\pi_*\mathcal{E}$.

We're in odd dimensions. . . what is this?

To define the index gerbe we take our surjective submersion $U \to B$ an open cover of B. This open cover is chosen such that D_0 can be deformed slightly

$$\mathsf{D}_\alpha = \mathsf{D}_0 + h_\alpha(\mathsf{D}_0)$$

via $h_\alpha \in C_c^\infty(\mathbb{R})$ such that D_α is invertible over U_α. Taking $U = \sqcup_\alpha U_\alpha$, we now wish to provide line bundles with connection on nonempty double overlaps $U_{\alpha\beta}$. To do this we notice that the (pseudo)differential operator

$$\frac{\mathsf{D}_\beta}{|\mathsf{D}_\beta|} - \frac{\mathsf{D}_\alpha}{|\mathsf{D}_\alpha|}$$

has, over any point $b \in B$, only 0 and ± 2 as its eigenvalues. Write $\mathsf{pr}_\pm$ for the projections to the ± 2 eigenspaces of $\pi_*\mathcal{E}$. It turns out that the images of these projections are finite-rank vector bundles over $U_{\alpha\beta}$. We can thus define the complex line bundle over $U_{\alpha\beta}$

$$L_{\alpha\beta} = \Lambda^{\mathrm{top}}(\mathsf{pr}_+\pi_*\mathcal{E}) \otimes \Lambda^{\mathrm{top}}(\mathsf{pr}_-\pi_*\mathcal{E})^{-1}.$$

This line bundle inherits a connection from the connection on $\pi_*\mathcal{E}$ (first induce a connection on the projection as $\mathsf{pr} \circ \nabla \circ \mathsf{pr}$ and then take appropriate exterior powers). It remains to construct the isomorphism μ, which is equivalent to giving a trivialization of $L_{\beta\gamma} \otimes L_{\alpha\gamma}^{-1} \otimes L_{\alpha\beta}$ over $U_{\alpha\beta\gamma}$. There is an obvious such trivialization arising from the following observation. Over the open $U_{\alpha\beta\gamma}$ we have

$$\mathsf{pr}_+ = \mathsf{pr}_{\alpha=-,\beta=+} = \mathsf{pr}_{\alpha=-,\beta=+,\gamma=+} \oplus \mathsf{pr}_{\alpha=-,\beta=+,\gamma=-}.$$

In other words, the eigenspace on which $\mathsf{D}_\beta/|\mathsf{D}_\beta|$ acts as $+1$ and $\mathsf{D}_\alpha/|\mathsf{D}_\alpha|$ acts as -1 splits into a direct sum of spaces where additionally $\mathsf{D}_\gamma/|\mathsf{D}_\gamma|$ acts as ± 1. This yields a decomposition

$$L_{\alpha\beta} \cong \Lambda^{\mathrm{top}} H_{-++} \otimes \Lambda^{\mathrm{top}}(H_{-+-}) \otimes \Lambda^{\mathrm{top}}(H_{+-+})^{-1} \otimes \Lambda^{\mathrm{top}}(H_{+--})^{-1}$$

where $H_{\alpha\beta\gamma}$ is the eigenspace on which the operators act by multiplication according to the given sign. A similar decomposition holds for $L_{\beta\gamma}$ and $L_{\alpha\gamma}$, from which it is easy to see that the triple required triple tensor product is canonically trivial. Moreover one can check the associativity condition required on quadruple overlaps.

To complete the construction of the index gerbe it remains to specify 2-forms B_α over each U_α such that the difference $B_\beta - B_\alpha$ on $U_{\alpha\beta}$ is equal to the curvature of the connection on $L_{\alpha\beta}$. This is where the eta invariant appears explicitly. Introduce a formal odd variable σ and define the rescaled Bismut superconnection

$$\mathbb{A}_{\alpha,s} = s\sigma\mathsf{D} + \nabla^{\pi_*\mathcal{E}} + \frac{1}{4s}\sigma c(T)$$

on $\pi_*\mathcal{E}$ restricted to U_α. Here $c(T)$ is Clifford multiplication by the curvature of the connection on $M \to B$. If we write tr_σ for the operator that projects onto coefficients of σ and then takes the trace, one can use methods from the proof of the local family index theorem to show that

$$\mathrm{tr}_\sigma\left(\frac{d\mathbb{A}_{\alpha,s}}{ds}e^{-\mathbb{A}^2_{\alpha,s}}\right)$$

has a nice enough asymptotic expansion for $s \to 0$ such that it makes sense to define

$$\tilde{\eta}_\alpha = \mathrm{f.p.}_{t\to 0}\int_t^\infty \mathrm{tr}_\sigma\left(\frac{d\mathbb{A}_{\alpha,s}}{ds}e^{-\mathbb{A}^2_{\alpha,s}}\right)ds$$

an even-degree inhomogeneous differential form on U_α. Here we are taking a suitably defined finite part of an otherwise overall divergent quantity (what BGV call a renormalized limit). After some detailed computations, one find that (up to some slightly finicky normalizations)

$$(\tilde{\eta}_\beta - \tilde{\eta}_\alpha)_{[2]} = F_{\alpha\beta},$$

where $F_{\alpha\beta}$ is the curvature of $L_{\alpha\beta}$ and [2] represents taking the two-form component. We conclude that the degree two components of the eta-forms yield the curving for Lott's index gerbe, which completes the outline of the construction (up to checking that the choices made in the construction are immaterial). The curvature of this gerbe, by the index theorem, turns out be the three-form component of the Chern character of the family index bundle.

The following example is more naturally defined as a principal bundle gerbe, so we will not go into as much detail, since we have been focusing on gerbes defined by line bundles.

Example 9 (Lifting bundle gerbe)**.** A certain class of degree 3 cohomology classes, and hence gerbes, arise naturally as obstructions to lifting the structure group of a principal bundle along a central extension. In particular let

$$1 \to U(1) \to \hat{G} \xrightarrow{t} G \to 1$$

be a central extension, i.e. $U(1) \subset Z(\hat{G})$. Suppose we are given a principal G-bundle $P \to X$. A lifting of structure group to $\hat{G}$ is the data of a principal $\hat{G}$-bundle $\hat{P} \to X$ together with a bundle map $\phi : \hat{P} \to P$ such that

$$\phi(\hat{p}\cdot\hat{g}) = \phi(\hat{p})\cdot t(\hat{g}).$$

The existence of a such a lift is given by a class in $H^2(X, U(1))$ as can be checked via Čech methods. Fix a good open cover $\mathcal{U}$ and denote by $g_{\alpha\beta} : U_{\alpha\beta} \to G$ the transition functions of P. As the opens $U_{\alpha\beta}$ are contractible, the $g_{\alpha\beta}$ can be lifted (the $U(1)$-bundle $\hat{G} \to G$ is trivial over $U_{\alpha\beta}$ so we need only choose a section) to $\hat{G}$, call them $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to \hat{G}$. Notice that although $g_{\alpha\beta}$ satisfies the cocycle condition, the lift $\hat{g}_{\alpha\beta}$ need not. Define

$$\varepsilon_{\alpha\beta\gamma} = \hat{g}_{\beta\gamma}\hat{g}^{-1}_{\alpha\gamma}\hat{g}_{\alpha\beta}$$

and note that $t(\varepsilon_{\alpha\beta\gamma}) = 1$ whence $\varepsilon_{\alpha\beta\gamma}$ is a $U(1)$-valued 2-cochain. A simple computation reveals that

$$\begin{aligned}(\delta\varepsilon)_{\alpha\beta\gamma\delta} &= \varepsilon_{\beta\gamma\delta}\varepsilon^{-1}_{\alpha\gamma\delta}\varepsilon_{\alpha\beta\delta}\varepsilon^{-1}_{\alpha\beta\gamma} \\ &= 1.\end{aligned}$$

whence ε defines a Čech cocycle for a class in $H^2(X;\underline{U(1)})$. Changing our choice of lifts $\hat{g}_{\alpha\beta}$ to $\hat{g}'_{\alpha\beta}$ is harmless, as it changes this cocycle by a coboundary: we can write $\hat{g}'_{\alpha\beta} = \hat{g}_{\alpha\beta}h_{\alpha\beta}$ for $h_{\alpha\beta} : U_{\alpha\beta} \to U(1)$. Then, since $U(1) \subset Z(\hat{G})$,

$$\varepsilon'_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}h_{\beta\gamma}h^{-1}_{\alpha\gamma}h_{\alpha\beta}$$

which differs from $\varepsilon_{\alpha\beta\gamma}$ by the coboundary δh. We conclude that the bundle P lifts to $\hat{P}$ if ε is trivial. Indeed, it is easy to see that we may as well assume $\varepsilon = 1$, whence the $\hat{g}_{\alpha\beta}$ define a principal $\hat{G}$-bundle constructed as a quotient of the disjoint union $\coprod_\alpha U_\alpha \times \hat{G}$. The morphism $t : \hat{G} \to G$ yields a map $\coprod_\alpha U_\alpha \times \hat{G} \to \coprod_\alpha U_\alpha \times G$ that descends to quotients because $t(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$.

Notice that from the short exact sequence of sheaves of abelian groups

$$0 \to \mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{\exp(2\pi i\cdot)} U(1) \to 0$$

(where $U(1)$ means the sheaf of $U(1)$-valued functions) and the fact that Ω^0 is fine (admits partitions of unity) and thus has no higher cohomology, we deduce that $H^2(X;U(1)) \cong H^3(X;\mathbb{Z})$. Let us construct a bundle gerbe corresponding to the class ε (really we will be constructing a principal bundle gerbe). The surjective submersion is given by the map $\pi : P \to X$. To obtain a $U(1)$-bundle Q over $P^{[2]}$ we consider $\hat{G} \to G$ as a principal $U(1)$-bundle and pull it back along the map $g : P^{[2]} \to G$ defined by

$$p \cdot g(p,p') = p',$$

i.e. we have

$$\begin{array}{ccc} Q & \longrightarrow & \hat{G} \\ \downarrow & & \downarrow{\scriptstyle t} \\ P^{[2]} & \xrightarrow{g} & G \end{array}$$

The isomorphism

$$\mu : \pi^*_{01}Q \otimes \pi^*_{12}Q \xrightarrow{\sim} \pi^*_{02}Q$$

Finish this

is the data of

2.3. **Categorical aspects.**

References