FACTORIZATION HOMOLOGY

Contents

Date: Fall 2017.

These are notes from John Francis' "Factorization homology" course taught during the Fall quarter of the 2017 year at Northwestern. Errors and inaccuracies are, as usual, due to the notetaker(s).

1. What is factorization homology? [09/20/17]

1.1. **Introduction.** What is factorization homology? Well, if it were an animal, I could describe it in two ways: distribution and phylogeny. More specifically, we will first see how factorization homology is distributed over the face of the planet. Then we will describe how it evolved from single-celled organisms, i.e. how you might come up with it yourself.

For the moment you can think of factorization homology as a sort of

generalized (co)sheaf homology.

Notice that this phrase can be hyphenated in two different ways. In one sense it is a generalization of the ideas of sheaf cohomology, and in the other it is a homology theory for generalized sheaves (or sheaf-like objects). In particular, factorization homology is a machine that takes two inputs: a geometry M and an algebraic object A. The output is

$$\int_M A,$$

the factorization homology of with coefficients in A.

1.2. **Examples.** Let's look at the first description: what are some examples of factorization homology that appear naturally in mathematics?

(1) **Homology.** Here M is a topological space and A is an abelian group. In this case the output is a chain complex

$$\int_M A \simeq \mathrm{H}_\bullet(M, A),$$

quasiisomorphic to singular homology with coefficients in A.

John: What is a theorem you can't prove without ordinary homology?

(2) **Hochschild homology.** Here M is a one-dimensional manifold – let's take in particular $M = S^1$ – and A will be an associative algebra. In this case

$$\int_{S^1} A \simeq \mathrm{HH}_\bullet A,$$

the Hochschild homology of A. You might be less familiar with this algebraic object than ordinary homology. It's importance comes from how it underlies trace methods in algebra (e.g. characteristic 0 representations of finite groups). Hochschild homology is a recipient of "the universal trace" and hence an important part of associative algebra. Note that $\mathrm{HH}_0 A = A/[A, A]$.

(3) **Conformal field theory.** This is in some sense the real starting point for the ideas we will develop in this class. Here M is a smooth complete etc. algebraic curve over $\mathbb{C}$ and A is a vertex algebra. In this case the output $\int_M A$ was constructed by Beilinson and Drinfeld, and is known as chiral homology of M with coefficients in A. It is a chain complex, with $\mathrm{H}_0(\int_M A)$ being the space of conformal blocks of the conformal field theory.

(4) **Algebraic curves over $\mathbb{F}_q$.** Here M is an algebraic curve over $\mathbb{F}_q$ and G is a connected algebraic group over $\mathbb{F}_q$. In this case $\int_M G$ is known as the Beilinson-Drinfeld Grassmannian and is a stack. One interesting property that it has is that

$$\mathrm{H}_\bullet\left(\int_M G, \bar{\mathbb{Q}}_\ell\right) \simeq \mathrm{H}_\bullet(\mathrm{Bun}_G(M), \bar{\mathbb{Q}}_\ell),$$

where here we are taking ℓ-adic cohomology. Although the Beilinson-Drinfeld Grassmannian is more complicated than the stack of principal G-bundles, it is more easily manipulated. We note that the equivalence above is a form of nonabelian Poincaré duality.

In particular, one might be interested in computing

$$\chi(\mathrm{Bun}_G(M)) = \sum_{[P]} \frac{1}{|\operatorname{Aut}(P)|},$$

which makes sense over a finite field. The computation of this quantity is known as Weil's conjecture on Tamagawa numbers.

(5) **Topology of mapping spaces.** Now M is an n-manifold without boundary and A will be an n-fold loop space, $A = \Omega^n Z = \mathrm{Maps}((D^n, \partial D^n), (Z, *))$. The output is a space weakly homotopy equivalent to $\mathrm{Maps}_c(M, Z)$ if $\pi_i Z = 0$ for $i < n$. This is also known as nonabelian Poincaré duality. Again the left hand side is more complicated but more easily manipulated.

(6) **n-disk algebra (perturbative TQFT).** Here M is an n-manifold and A is an n-disk algebra (or an E_n-algebra) in chain complexes. The output is a chain complex and has some sort of interpretation in physics. One thinks of A as the algebra of observables on $\mathbb{R}^n$, and $\int_M A$ is the global observables (in some derived sense). In a rough cartoon of physics, one assigns to opens sets of observables, and a way to copmute expectation values. Factorization homology puts together local observables to global observables:

$$\mathrm{Obs}(M) \simeq \int_M A,$$

at least if we are working in perturbative QFT.

(7) **TQFT.** Here M is an n-manifold (maybe with a framing) and A is an (∞, n)-category (enriched in $\mathcal{V}$). The output is a space (if enriched, an object of $\mathcal{V}$), which is designed to remove the assumptions from the examples above.

That's all the examples for now. Next class we'll go over how one might have come up with factorization homology. It is worth noting that in this class we will focus on learning factorization homology as a **tool** instead of aiming to reach some fancy theorem. Hopefully this will teach you how to apply it in contexts you might be interested in.

Pax: What is the physical interpretation of the first and second chiral homologies? **John:** One might be interested in things like Wilson lines, where these higher homology groups come into play.

2. How to come up with factorization homology yourself [09/22/17]

2.1. **Kan extensions.** Consider the following thought experiment. Suppose you want to study objects in some context $\mathcal{M}$. Unfortunately objects here are pretty hard in general. Inside $\mathcal{M}$, however, we have some objects $\mathcal{D} \subset \mathcal{M}$ that are particulary simple, and moreover we know that everything else in $\mathcal{M}$ is "built out of" objects in $\mathcal{D}$.

Let's consider the example where $\mathcal{M}$ is a nice category of (homotopy types of) topological spaces. Let $\mathcal{D}$ consist of the point, i.e. all contractible spaces. Now to study $\mathcal{M}$ we might map functors out of it into some category $\mathcal{V}$. Let's start with $\mathcal{D}$ instead. Consider $\mathrm{Fun}(*, \mathcal{V})$. Of course this is canonically just $\mathcal{V}$. How do we extend this to studying $\mathcal{M}$? We have an obvious restriction map

$$\mathrm{Fun}(\mathcal{M}, \mathcal{V}) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(*, \mathcal{V}).$$

We want to look for a left adjoint to this functor ev_* There are two different things we could do. We could ignore the homotopy-ness of everything, and take the naive categorical left-adjoint. If, say $\mathcal{V}$ is the category of chain complexes, this naive left-adjoint produces a stupid answer... depending on what our precise definitions are. Let's suppose that by $\mathcal{M}$ we meant the homotopy category of spaces (here objects are spaces and maps are sets of homotopy classes of maps). Then we are extending

John: If you don't know what a left adjoint is you should learn it because I won't tell you. No, I'm not joking (laughs).

$$\begin{tikzcd} * \arrow[r, "A"] \arrow[d] & \mathcal{V} \\ \mathsf{hoSpaces} \arrow[ur, dashed] & \end{tikzcd}$$

A naive left adjoint would take the functor A to the functor sending a space X to the stupid answer $A^{\oplus \pi_0 X}$ (on morphisms take summands to summands corresponding to where the connected components are sent). Similarly if we take $\mathcal{M}$ to be just spaces and all continuous maps, X would be sent to $A^{\oplus X}$. Here by X we mean the underlying set of elements of X.

Why is this a left adjoint?

There is a more sophisticated notion of a derived or homotopy left adjoint. Suppose now that by $\mathcal{M}$ we mean the topological category of spaces, where the mapping sets are spaces equipped with the compact-open topology. Now we take a *homotopy* Kan extension. This fancy left adjoint will now send a space X to the the chain complex $C_*(X, A)$ (up to equivalence). Hence we see that we can recover homology from this paradigm of extending a simpler invariant to the whole category.

How do we choose what $\mathcal{D}$ and $\mathcal{M}$ are? Suppose we want to study $\mathcal{F}(M)$ for $M \in \mathcal{M}$. For concreteness, let's say we're studying manifolds. The most basic question to ask: is there a local-to-global principle for $\mathcal{F}$? The simplest case is for $\mathcal{F}$ to be a sheaf, i.e.

$$\mathcal{F}(M) \xrightarrow{\sim} \lim_{U \in \mathcal{U}} \mathcal{F}(U).$$

If so you don't need factorization homology, and you can just leave.

For instance, consider $\mathcal{F} = C_*(\mathrm{Maps}(\cdot, Z))$ taking spaces to chain complexes. Is this a sheaf? Well if we forget about C_*, we get a sheaf, as a map into Z is the same as giving maps on subsets of the domain that agree on overlaps. What does taking chains do? Well notice that

$$\begin{aligned} C_*(\mathrm{Maps}(U \coprod V, Z)) &= C_*(\mathrm{Maps}(U, Z) \times \mathrm{Maps}(V, Z)) \\ &= C_*(\mathrm{Map}(U, Z)) \otimes C_*(\mathrm{Maps}(V, Z)). \end{aligned}$$

Why? This is not a sheaf because in this case tensor products and direct sums are never the same for these chain complexes! So what can we do? We need to change what we consider $\mathcal{D}$ to be from open coverings to something else.

Idea: to study $\mathcal{F}$ maybe there are more general arrangements of $\mathcal{D} \subset \mathcal{M}$ such that local-to-global principles still apply, without $\mathcal{F}$ being a sheaf.

2.2. **Manifolds.** The following problem will guide us for the next few weeks.

> Let M be a manifold and let Z be a space. Calculate the homology of the mapping space $\mathrm{H}_\bullet \mathrm{Maps}(M, Z)$.

To begin, let us specify which categories we will be working with.

Definition 1. Let Mfld_n be the (ordinary) category of smooth n-manifolds, with $\mathrm{Hom}(M, N) = \mathrm{Emb}(M, N)$ the set of smooth embeddings of M into N. Similarly, let $\mathcal{M}\mathsf{fld}_n$ be the *topological* category of smooth n-manifolds, with $\mathrm{Hom}(M, N) = \mathrm{Emb}(M, N)$ the *space* of smooth embeddings of M into N, equipped with the compact open smooth topology.

The compact open smooth topology takes a bit of work to define, so we'll leave that as background reading. A good reference is Hirsch's book on differential topology [Hir94]. Roughly, convergence in this topology is pointwise in the map as well as all its derivatives. To get a feel for what this entails, consider a knot. Locally tighten the knot until the knot turns (locally) into a line. These knots would would converge in the usual compact-open topology to another knot, but in the smooth topology, they do not converge as the tightening procedure creates sharp kinks. In particular $\pi_0 \mathrm{Emb}(S^1, \mathbb{R}^3)$ is very different from $\pi_0 \mathrm{Emb}^{\mathrm{top}}(S^1, \mathbb{R}^3)$.

Definition 2. We define the category Disk_n to be the full subcategory of Mfld_n where the objects are finite disjoint unions of standard Euclidean spaces $\coprod_I \mathbb{R}^n$. Similarly the category $\mathcal{D}\mathsf{isk}_n$ is the full *topological* subcategory of $\mathcal{M}\mathbf{fld}_n$ where the objects are finite disjoint unions of Euclidean space.

Observe that $\mathrm{Hom}_{\mathcal{D}\mathsf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) = \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 3. *The map* $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \to GL_n(\mathbb{R}) \simeq O_n\mathbb{R}$ *given by differentiating at the origin is a homotopy equivalence.*

Proof sketch. There is an obvious map $GL_n\mathbb{R} \to \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$. One of the composites is thus clearly the identity. It remains to show that the other composition is homotopic to the identity. The homotopy is given by shrinking the embedding down to zero. □

This fact should fill you with hope. The objects which are building blocks of manifolds have automorphism spaces that are, up to homotopy, just finite-dimensional manifolds. Actually it will be useful to think of the n-disks as some sort of algebra.

Definition 4. An n-disk algebra in $\mathcal{V}$ is a symmetric monoidal functor $A : \mathcal{D}\mathsf{isk}_n \to \mathcal{V}$.

As we stated before, our first goal in this class is to understand the homology $\mathrm{H}_* \operatorname{Maps}(M, Z)$ using n-disk algebras and factorization homology.

Question from someone: what's the relation with E_n-algebras? John: It turns out that E_n-algebras are equivalent to n-disk algebras with framing.

Question from Tochi: what if you work with manifolds with boundary? John: well if you require boundaries to map to boundaries you can make the same definitions. You then have to work with Euclidean spaces and half-spaces. You'll end up with two types of algebras instead of just n-disk algebras.

3. Framings [09/25/17]

3.1. Framed embeddings, naively.

Definition 5. A **framing** of an n-manifold M is an isomorphism of vector bundles $TM \cong M \times \mathbb{R}^n$.

Of course, not all manifolds have framings. For instance, one can check that all (compact oriented) two-manifolds except for $S^1 \times S^1$ do not admit framings. You might use the Poincaré-Hopf theorem, which expresses the Euler characteristic as a sum of the index of the zeroes of a vector field v on M that has isolated zeroes. Hence if M is framed, the Euler characteristic of M must be zero.

Here is an example of a theorem that John does not know how to prove without the use of homology.

Theorem 6 (Whitney or Wu)**.** *Every orientable three-manifold admits a framing.*

Pax: isn't there a later proof of this via geometric methods by Kirby? John: well ok I don't know how to prove it without homology...

Notice that any Lie group has a framing, as one takes a basis for the Lie algebra and pushes it forward by the group action. On the other hand, manifolds of dimension four generally do not have framings (at least in John's experience).

We can ask the following question: what is a framed open embedding? There are a few options. The naive (strict) option is as follows. Suppose that we have an open embedding $M \hookrightarrow N$ of framed manifolds. The pullback of TN is TM, we have two different trivializations of TM. We might ask that the induced map of trivial bundles $M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ be the identity. In other words, we ask the two framings to be the same.

Okay fine, but lets think about what we want the answer to be. Embeddings are very flexible you can stretch them and twist them. But strict framed embeddings are very rigid the way we've defined them above. For instance, they are automatically isometries (giving the fibers the usual Euclidean metric). But of course there aren't very many isometric embeddings into a compact manifold. Thus the strict definition of a framed embedding is not what we want to work with.

3.2. Framed embeddings, homotopically.

Let's consider a more lax definition. Thinking homotopy theoretically, recall that the tangent bundle is classified by a map $TM : M \to \mathrm{Gr}_n \mathbb{R}^\infty$. This map is of course defined only up to homotopy. That's fine, just choose a representative. Over the infinite Grassmannian we have the infinite Stiefel manifold $V_k(\mathbb{R}^\infty) \to \mathrm{Gr}_n \mathbb{R}^\infty$. Choosing a lift

$$\begin{tikzcd} & V_n\mathbb{R}^\infty \arrow[d] \\ M \arrow[ur, dashed, "\phi_M"] \arrow[r, "TM"'] & \mathrm{Gr}_n \mathbb{R}^\infty \end{tikzcd}$$

Why?

is precisely the data of a framing. Suppose now that we have an embedding $M \hookrightarrow N$ where M, N are framed by ϕ_M and ϕ_N respectively. The lax definition of a framed embedding is now going to be extra data: an embedding together with a homotopy between the framings ϕ_M and $\phi_N|_M$.

Definition 7. The space of framed embeddings $\mathrm{Emb}^{fr}(M,N)$ is the homotopy pullback

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(M,N) \arrow[r] \arrow[d] & \mathrm{Emb}(M,N) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(M,N) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\,\mathbb{R}^\infty}(M,N)
\end{tikzcd}$$

In particular a framed embedding is an embedding $M \hookrightarrow N$ and a homotopy in $\mathrm{Map}_{\mathrm{Gr}_n\,\mathbb{R}^\infty}(M,N)$ between the images along each map.

Exercise 8. Check that $V_n\mathbb{R}^\infty \simeq *$.

With all this talk of homotopy pullbacks (which we'll talk about in more detail next time) it looks like we've made things more complicated, whereas we introduced framings to make things simpler. Let's calculate $\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n)$ as an example. By definition, this sits in the following diagram

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(\mathbb{R}^n,\mathbb{R}^n) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\,\mathbb{R}^\infty}(\mathbb{R}^n,\mathbb{R}^n).
\end{tikzcd}$$

Notice that the bottom left object is homotopy equivalent to $\mathrm{Maps}_*(\mathbb{R}^n,\mathbb{R}^n) \simeq *$. The bottom right space is homotopy equivalent to the loop space $\Omega\,\mathrm{Gr}_n\,\mathbb{R}^\infty \simeq \Omega BO(n) \simeq O(n)$. From last time, $\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n) \simeq \mathrm{Diff}(\mathbb{R}^n) \simeq GL(n) \simeq O(n)$ (this is **homework 1**). Now the vertical map on the right is a homotopy equivalence. This implies (by some machinery) that the vertical map on the left is an equivalence. We conclude that

$$\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n) \simeq *.$$

The rest of **homework 1** is to show that $\mathrm{Emb}(\mathbb{R}^n,N)$ is homotopy equivalent to the frame bundle of TN. Applying this to the diagram above where we replace the second copy of $\mathbb{R}^n$ with N, we obtain

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\mathbb{R}^n,N) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^n,N) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(\mathbb{R}^n,N) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\,\mathbb{R}^\infty}(\mathbb{R}^n,N).
\end{tikzcd}$$

Now the same argument will show that the vertical map on the right is an equivalence, and that the map of the left is an equivalence. It follows now that

$$\mathrm{Emb}^{fr}(\mathbb{R}^n,N) \simeq N.$$

Hence we see that by adding framings we are replacing the role of the orthogonal group by that of a point. Indeed, this will allow for an easier transition between algebra and topology.

Definition 9. We define the category $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}$ to be the topological category consisting of finite disjoint unions of open unit disks $\coprod_I D$ under rectilinear embeddings. In other words, embeddings which can be written as a composition of translations and dilations. Here we use the usual topology indcued from the smooth compact-open topology.

One advantage of rectilinear embeddings is that they are easy to analyze. For instance, the space of embeddings from a single disk to a single disk is contractible: take an embedding, translate it to the origin, and the expand it outwards. In this way $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(D, D) = \mathrm{Emb}^{\mathrm{rect}}(D, D)$ deformation retracts onto the identity map. More generally, one checks that there is a homotopy equivalence

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(\coprod D^n, D^n) \xrightarrow{\sim} \mathrm{Conf}_k(D^n)$$

Next time we will prove the following.

Proposition 10. *There is a homotopy equivalence* $\mathscr{D}\mathsf{isk}_n^{rect} \simeq \mathscr{D}\mathsf{isk}_n^{fr}$.

4. Homotopy pullbacks and framing [09/27/17]

Let's define more precisely some of the terms we used last time.

4.1. Homotopy pullbacks.

Definition 11. Suppose we have a map $f : X \to B$ together with a point $* \in B$. The homotopy fiber of $X \to B$ over $* \in B$ is the fiber product

$$\mathrm{hofiber}(f : X \to B) := \{*\} \times_B \mathrm{Maps}([0,1], B) \times_B X.$$

In particular it is the space of triples $(*, \phi, x)$ where $\phi(0) = *$ and $\phi(1) = f(x)$.

$$\begin{tikzcd} \mathrm{hofiber}(f) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

Lemma 12. *The formation of homotopy fibers is homotopy invariant. More precisely, given an weak equivalence of spaces $X \to X'$ over B a pointed space via maps f and g,*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & X' \arrow[dl] \\ & B & \end{tikzcd}$$

then the homotopy fiber of f is weakly equivalent to the homotopy fiber of g.

Proof. Simply apply the (naturality of the) long exact sequence on homotopy groups for a Serre fibration to the map of fibrations

Why are these fibrations?

$$\begin{tikzcd} \mathrm{hofiber}(f) \arrow[rr] \arrow[d] & & \mathrm{hofiber}(g) \arrow[d] \\ \mathrm{Maps}([0,1], B) \times_B X \arrow[rr] \arrow[dr] & & \mathrm{Maps}([0,1], B) \times_B X' \arrow[dl] \\ & B & \end{tikzcd}$$

We conclude that $\pi_* \mathrm{hofiber}(f) \cong \pi_* \mathrm{hofiber}(g)$. □

Homework 2: Show, more generally, that homotopy pullbacks are homotopy invariant.

Recall last time we were discussing $\mathrm{Maps}_B(M, N)$ for some space B: maps "over" B. This object is defined to be the homotopy

$$\begin{tikzcd} \mathrm{Maps}_B(M, N) \arrow[r] \arrow[d] & \mathrm{Maps}(M, N) \arrow[d] \\ * \arrow[r] & \mathrm{Maps}(M, B) \end{tikzcd}$$

In our case the map on the bottom is (a choice of) the map classifying the tangent bundle of M. Returning to last lecture, notice that by homotopy invariance we can argue that $\mathrm{Maps}_{V_n\mathbb{R}^\infty}(M, N) \simeq \mathrm{Maps}(M, N)$ since $V_n\mathbb{R}^\infty \simeq *$. Hopefully this background fills in some of the gaps we left open during last lecture.

But here we are using homotopy invariance in the base?

What is a homotopy equivalence of topological categories?

4.2. **Framed vs rectilinear n-disks.** Let us now return to our assertion from last time.

Proposition 13. *There is a functor $\mathscr{D}\mathsf{isk}_n^{rect} \to \mathscr{D}\mathsf{isk}_n^{fr}$ which is a homotopy equivalence.*

Proof. Using the computations from last lecture we see that

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}(\mathbb{R}^n, \mathbb{R}^n) \simeq * \simeq \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(D^n, D^n).$$

What this functor does on objects is clear. On morphisms, the framing is determined by the dilation factor present in the rectilinear embeddings. More generally, consider

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_I D^n, \coprod_J D^n\right) = \coprod_{\pi: I \to J} \prod_J \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_{\pi^{-1}(j)} D^n, D^n\right).$$

So it suffices to show that

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}\left(\coprod_I \mathbb{R}^n, \mathbb{R}^n\right) \simeq \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_I D^n, D^n\right).$$

Recall that $\mathrm{ev}_0 : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(\coprod D^n, D^n) \to \mathrm{Conf}_I(D^n)$ is a homotopy equivalence, which we mentioned ast time. Returning to our homotopy pullback square

$$\begin{tikzcd}
\mathrm{Emb}^{rect}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Emb}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[d] \\
* \simeq \mathrm{Maps}_{EO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] & \mathrm{Maps}_{BO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n)
\end{tikzcd}$$

notice that

$$\begin{tikzcd}
\mathrm{Fr}(TM) \simeq \mathrm{Emb}(\mathbb{R}^n, M) \arrow[d, "\mathrm{ev}_0"] & \mathrm{Emb}((\mathbb{R}^n, 0), (M, x)) \simeq O(n) \arrow[l] \arrow[d] \\
M & \{x\} \arrow[l]
\end{tikzcd}$$

Likewise

$$\begin{tikzcd}
\mathrm{Emb}(\coprod \mathbb{R}^n, M) \arrow[d, "\mathrm{ev}_0"] & \prod_I O(n) \arrow[l] \arrow[d] \\
\mathrm{Conf}_I(M) & \{x_1, \ldots, x_I\} \arrow[l]
\end{tikzcd}$$

Hence $\mathrm{Maps}_{BO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n) \simeq \prod_I \mathrm{Maps}_{BO(n)}(\mathbb{R}^n, \mathbb{R}^n) \simeq \prod_I O(n)$.

Up to homotopy, we now obtain

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Conf}_I(\mathbb{R}^n) \times \prod_I O(n) \arrow[d] \\
* \arrow[r] & \prod_I O(n)
\end{tikzcd}$$

so we conclude that $\mathrm{Emb}^{fr}(\coprod \mathbb{R}^n, \mathbb{R}^n) \simeq \mathrm{Conf}_I(\mathbb{R}^n)$ which concludes the proof of the proposition. $\square$

Example 14. Consider the case $n = 1$. What do the framed and rectilinear embeddings look like in this case? Well $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}(\coprod_I \mathbb{R}^1, \mathbb{R}^1) \simeq \mathrm{Conf}_I(\mathbb{R}^1)$ is discrete up to homotopy, and thus identified noncanonically with the symmetric group on I letters.

Recall a definition from the first day.

Definition 15. An $\mathcal{E}_n$ algebra in $\mathcal{V}$ is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathcal{V}^{\otimes}$.

Next time we will see that $\mathcal{E}_1$-algebras are, in a suitable sense, equivalent to associative algebras.

5. Examples of n-disk algebras [09/29/2017]

Why is this?

Notice that we have a functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}} \to \mathscr{D}\mathsf{isk}_n$. In particular, the former category has *less* structure than the latter.

Let's recall the following way of thinking about a commutative algebra.

Definition 16. A commutative algebra in $\mathcal{V}^{\otimes}$ (a symmetric monoidal category) is a symmetric monoidal functor

$$(\mathsf{Fin}, \coprod) \xrightarrow{A} (\mathcal{V}, \otimes),$$

where Fin is the category of finite sets.

This probably looks a little unfamiliar, so let's unpack it. Observe that the underlying object is $A = A(*)$. The unit morphism is $A(\varnothing) = 1_{\mathcal{V}} \to A(*)$. Here $1_{\mathcal{V}}$ is the symmetric monoidal unit in $\mathcal{V}$. The multiplicative structure comes from the map from the two-point set to the one-point set, and the commutativity follows from the fact that this map is Σ_2-invariant and that A is a *symmetric* monoidal functor so that $A^{\otimes 2} \to A$ is Σ_2-invariant as well.

Definition 17. For $\mathcal{V}$ a symmetric monoidal topological category, an **n-disk algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n \to \mathcal{V}$. Similarly a **framed n-disk algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}} \to \mathcal{V}$ and a **$\mathcal{E}_n$-algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathcal{V}$.

Today we will discuss examples of n-disk algebras for $\mathcal{V}$ being chain complexes and toplogical spaces.

(1) There are the trivial n-disk algebras. For instance, consider $A = \mathbb{Z}$, which sends

$$\textstyle\coprod_I \mathbb{R}^n \longrightarrow \mathbb{Z}^{\otimes I} \cong \mathbb{Z}$$

and any embedding

$$\textstyle\coprod_I \mathbb{R}^n \hookrightarrow \coprod_J \mathbb{R}^n \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}.$$

We can all agree that this is pretty trivial. More generally, we might take $A = \mathbb{Z} \oplus B$, which sends $\coprod_I \mathbb{R}^n$ to $(\mathbb{Z} \oplus B)^{\otimes I}$ and sends $\coprod_I \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ to a map $(\mathbb{Z} \oplus B)^{\otimes I} \to \mathbb{Z} \otimes B$. What is this map? Let's start by looking at the case where $|I| = 2$. In that case take the map

$$\mathbb{Z} \oplus \mathbb{Z} \otimes B \oplus B \otimes \mathbb{Z} \oplus B \otimes B \xrightarrow{\mathrm{id}_{\mathbb{Z}} \oplus \mathrm{id}_B \oplus \mathrm{id}_B \oplus 0} \mathbb{Z} \oplus B.$$

This map looks weird. Fix it.

You can generalize this for larger I – just take the product on the B factors to be zero.

(2) Now let $A : (\mathsf{Fin}, \coprod) \to (\mathsf{Ch}, \otimes)$ be a commutative dg algebra. There is a natural symmetric monoidal functor $\pi_0 : (\mathscr{D}\mathsf{isk}_n, \coprod) \to (\mathsf{Fin}, \coprod)$ which sends $\coprod_I \mathbb{R}^n \mapsto \pi_0(\coprod_I \mathbb{R}^n) = I$. The composition of these maps gives us an n-disk algebra. The idea here is that in an n-disk algebra there is not just one way of multiplying things. Indeed, there are $\mathrm{Emb}(\coprod_2 \mathbb{R}^n, \mathbb{R}^n)$ multplications. What we have just done is used the π_0 functor to reduce these various multiplications into the unique multiplication coming from the unique map from the two-point set to the one-point set.

(3) The next example is that of an n-fold loop space of a pointed space $(Z, *)$. We will construct a functor $\mathscr{D}\mathsf{isk}_n \to \mathsf{Top}$ and then postcompose with C_* to obtain a chain complex. This first functor is $\Omega^n Z : \mathscr{D}\mathsf{isk}_n \to \mathsf{Top}$, which we will now define. Recall that for M a space and Z a pointed space, we say that a map $M \to Z$ is **compactly supported** if there exists $K \subset M$ with K compact and such that $g|_{M\setminus K} = * \in Z$. Then we define

$$\Omega^n Z := \mathrm{Maps}_c(-, Z) : (\mathscr{D}\mathsf{isk}_n, \coprod) \to (\mathsf{Top}, \times).$$

If you haven't thought much about compactly supported maps then there is something you have to check. Observe that if

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d] & Z \\ V \arrow[ur, dashed] & \end{tikzcd}$$

the map $U \hookrightarrow V$ is an open embedding then the map $\bar{g}$, given by sending a point v to $g(v)$ for $v \in U$ and $*$ otherwise, is continuous (**homework 3**). Hence Maps_c is covariant via this extension by zero procedure. Moreover it is symmetric monoidal as it sends disjoint unions to products.

Why is this called the n-fold loop space? Well notice that

$$\begin{aligned} \Omega^n Z &= \mathrm{Maps}((D^n, \partial D^n), (Z, *)) \\ &\simeq \mathrm{Maps}_c(\mathbb{R}^n, Z) \end{aligned}$$

where we identify $\mathbb{R}^n$ with the interior of the closed disk D^n. In total, we get

$$\mathscr{D}\mathsf{isk}_n \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Top} \xrightarrow{C_*} \mathsf{Ch}$$

whose composite we write $C_*\Omega^n Z$.

What happens if we don't use compactly supported and take values in cochains? What is this n-disk algebra in terms of things we know?

(4) At the opposite end of the spectrum from trivial algebras are free algebras. The **free $\mathcal{E}_n$ algebra** on $V \in (\mathsf{Ch}, \otimes)$, which we'll notate as

$$\mathcal{F}_{\mathcal{E}}(V) : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathsf{Ch},$$

sends

$$\mathbb{R}^n \mapsto \bigoplus_{k\geq 0} C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k}.$$

Here the Σ_k denotes the diagonal quotient. We will define what it does on morphisms in a moment.

This has the universal property that given any map of chain complexes $V \to A$ for A an $\mathcal{E}_n$-algebra (by this we mean a map of chain complexes $V \to A(\mathbb{R}^n)$), there exists a unique map of $\mathcal{E}_n$-algebras such that the diagram

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d] & A \\ \mathcal{F}_{\mathcal{E}_n}(V) \arrow[ur, dashed] & \end{tikzcd}$$

commutes. The vertical map $V \to \mathcal{F}_{\mathcal{E}_n}$ is given by the inclusion into the $k = 1$ summand which is just V.

What is this dashed map? For each k we need a map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k} \to A(\mathbb{R}^n).$$

To do this we use the map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k} \xrightarrow{\mu^{\otimes k}} C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} A^{\otimes k}$$

and then use the multiplication for A. Let's explain this. Notice that $A : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathsf{Ch}$ and we have $\mathrm{Emb}^{rect}(\coprod_k D^n, D^n) \to \mathrm{Maps}_{\mathsf{Ch}}(A^{\otimes k}(D^n), A(D^n))$ which by Dold-Kan (recall that everything is enriched in Top) corresponds to a map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}}(A^{\otimes k}(D^n), A(D^n))$$

that is Σ_k-equivariant. Because of the equivariance it factors to the quotient, which gives us the multiplication map. Now apply (equivariant) tensor-hom adjunction to obtain this multiplication map. Okay, but we haven't yet shown that the free thing is actually an $\mathcal{E}_n$-algebra, but we're out of time.

(5) The last example we were gonna talk about is pretty awesome. Too bad we're out of time.

Notice that if Z was an Eilenberg-MacLane space, there is overlap between examples 2 and 3.

6. Examples, continued [10/02/2017]

Last lecture we ran out of time in the proof of the following result.

Proposition 18. *The functor $\mathcal{F}_{\mathcal{E}_n}(V)$ sending*

$$D^n \mapsto \bigoplus_{k\geqslant 0} C_* \left(\mathrm{Emb}^{rect}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k}$$

is the free $\mathcal{E}_n$-algebra on $V \in \mathsf{Ch}$.

Proof. Last time we showed that this functor satisfied the correct universal property though we hadn't yet specified what it did on morphisms. To define what it does to morphisms we need to construct a map

$$\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n) \to \mathrm{Maps}_{\mathsf{Ch}}(\mathcal{F}_{\mathcal{E}_n}(V)^{\otimes I}, \mathcal{F}_{\mathcal{E}_n}(V)).$$

By the Dold-Kan correspondence this is equivalent to specifying a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \to \underline{Hom}(\mathcal{F}_{\mathcal{E}_n}(V)^{\otimes I}, \mathcal{F}_{\mathcal{E}_n}(V))$$

which by the tensor-(internal)hom adjunction, is equivalent to the data of a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \otimes \mathcal{F}_{\mathcal{E}_n} \to \mathcal{F}_{\mathcal{E}_n}(V),$$

in other words, a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \otimes \left(\bigoplus_{k\geqslant 0} C_*(\mathrm{Emb}(\coprod_k D^n, D^n)) \otimes_{\Sigma_k} V^{\otimes k}\right)^{\otimes I} \to \mathcal{F}_{\mathcal{E}_n}(V).$$

Let's maybe just look at the left hand side in the case where $I \cong \{0,1\}$:

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \otimes \bigoplus_{k_0,k_1} \left(C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0} D^n, D^n)) \otimes_{\Sigma_{k_0}} V^{\otimes k_0} \otimes C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_1} D^n, D^n)) \otimes_{\Sigma_{k_1}} V^{\otimes k_1}\right)$$

$$= \bigoplus_{k_0,k_1\geq 0} C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \times \mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0} D^n, D^n) \times \mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_1} D^n, D^n)\right) \otimes_{\Sigma_{k_0}\times\Sigma_{k_1}} V^{\otimes(k_0+k_1)}$$

But from this last expression it is easy to see now that we have a map from what's in the parentheses to $\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0+k_1} D^n, D^n) \otimes_{\Sigma_{k_0+k_1}} V^{\otimes(k_0+k_1)}$ by composing the embeddings (up to keeping track of the symmetric group). □

Let's talk about the example that we didn't have time to discuss at the end of last class. This is the class of $\mathcal{E}_n$ enveloping algebras of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. For simplicity we'll work over $\mathbb{R}$.

John: this works for Lie algebras valued in spectra too, up to some changes.

We define a functor $\mathscr{D}\mathsf{isk}_n \to \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Ch}_{\mathbb{R}})$ which sends $U \mapsto \Omega^*_c(U, \mathfrak{g})$, i.e. a Euclidean space to its space of compactly supported de Rham forms. Notice that this construction sends disjoint unions to direct sums. We now postcompose with the Chevalley complex C^{Lie}_* (or if you like $C^{\mathsf{Lie}}_*(\mathfrak{g}) \simeq \mathbb{R} \otimes^{\mathbb{L}}_{\mathcal{U}\mathfrak{g}} \mathbb{R}$). We will write this composite functor as $C^{\mathsf{Lie}}_*(\Omega^*_c(\bullet, \mathfrak{g}))$, and it sends disjoint unions to tensor products.

$$\mathscr{D}\mathsf{isk}_n \longrightarrow \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Ch}_{\mathbb{R}}) \longrightarrow \mathsf{Ch}_{\mathbb{R}}$$

We will use the fact that
$$C_*^{\mathsf{Lie}}(\mathfrak{g}\oplus\mathfrak{g}') \simeq C_*^{\mathsf{Lie}}(\mathfrak{g})\oplus C_*^{\mathsf{Lie}}(\mathfrak{g}').$$
We claim that for $n=1$,
$$C_*^{\mathsf{Lie}}(\Omega_c^*(\mathbb{R}^1,\mathfrak{g}))\simeq \mathcal{U}\mathfrak{g}.$$
In particular this functor which maps Lie algebras to $\mathcal{E}_n$-algebras is left-adjoint to the forgetful functor $\mathsf{Alg}_{\mathcal{E}_n}\to\mathsf{Alg}_{\mathsf{Lie}}$.

Here's a small aside. Where is this Lie algebra structure coming from? Well notice that we have a map
$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n))\otimes A^{\otimes 2}\to A.$$
But notice that the left-hand side is homotopic to $C_*(S^{n-1})$ (do this exercise!). At the level of homology, this gives a map $\mathrm{H}_* S^{n-1}\otimes_{\mathbb{R}}(\mathrm{H}_* A)^{\otimes 2}\to \mathrm{H}_* A$. There are two generators for the homology of S^{n-1} and so we a degree 0 map
$$\mathrm{H}_* A\otimes \mathrm{H}_* A\to \mathrm{H}_* A,$$
which is the associative algebra structure. However, we have another map coming from the fundamental class of S^{n-1},
$$(\mathrm{H}_* A\otimes \mathrm{H}_* A)[n-1]\to \mathrm{H}_* A,$$
is a Lie algebra structure on $\mathrm{H}_* A[1-n]$. (Everything here should be valued in Ch) A reference for this forgetful functor is a paper by F. Cohen.

Since we're almost out of time, let me give you a hint of what we'll be doing next. In factorization homology we are given some functor $A:\mathscr{D}\mathsf{isk}_n\to\mathsf{Ch}$ (or into Top). Factorization homology is an extension
$$\int_M A = \mathrm{hocolim}\;(\mathscr{D}\mathsf{isk}_{n/M}\xrightarrow{A}\mathsf{Ch})$$
an extension that fits into
$$\begin{tikzcd}
\mathscr{D}\mathsf{isk}_{n/M} \arrow[r, "A"] \arrow[d] & \mathsf{Ch} \\
\mathsf{Mfld}_n \arrow[ur, dashed] &
\end{tikzcd}$$

We need to define not only the homotopy colimit but also what we mean by $\mathscr{D}\mathsf{isk}_{n/M}$. What do we want it to be? Its mapping spaces should fit into the homotopy pullback diagram
$$\begin{tikzcd}
\mathrm{Maps}_{\mathscr{D}\mathsf{isk}_{n/M}}(U,V) \arrow[r] \arrow[d] & \mathrm{Emb}(U,V) \arrow[d, "V\hookrightarrow M"] \\
* \arrow[r, "U\hookrightarrow M"] & \mathrm{Emb}(U,M)
\end{tikzcd}$$

As usual, if we require this to be a pullback instead of a homotopy pullback this space will be too small. In fact, it will be empty. Okay, you say – so let's just define a category of n-disks with these mapping spaces. The problem that you will run into here is that the composition will be associative only up to homotopy due to the composition of the paths in $\mathrm{Emb}(U,M)$ required by the adjective "homotopy". So we'll have to dip our toes into the theory of infinity-categories, which neatly deals with both this issues and homotopy colimits.

7. Homotopy colimits [10/04/2017]

We are interested in proving the following result.

Theorem 19. *The homotopy colimit is homotopy invariant. More precisely, given two functors $F, G : \mathcal{C} \to \mathsf{Top}$ and any natural transformation $\alpha : F \implies G$ such that for all $c \in \mathcal{C}$, $\alpha(c) : F(c) \to G(c)$ is a homotopy equivalence, then*

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{C}} G$$

is a homotopy equivalence.

Notice that we can replace homotopy equivalence everywhere with weak homotopy equivalence. Actually we will sketch the proof. The details will be left as **homework 4.** There is a problem in the usual theory of colimits: they are not homotopy invariant. Consider the following simple example. We have a map of spans

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

where the vertical arrows are homotopy equivalences. But the colimits of the top and bottom rows are S^n and $*$ respectively, which are of course not homotopy equivalent.

We have two basic tools that we will use to fix this: Mayer-Vietoris and Seifert-van Kampen.

Lemma 20. *Mapping cones are homotopy invariant. More precisely, if we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

where the vertical arrows are homotopy equivalences then there is an induced homotopy equivalence on cones, $\operatorname{cone} f \simeq \operatorname{cone} f'$.

Proof. Recall that the cone is written as the colimit

$$\operatorname{cone} f = * \sqcup_{X \times \{0\}} X \times [0,1] \sqcup_{X \times \{1\}} Y.$$

Notice that we have maps $\operatorname{cyl} f \to \operatorname{cyl} f'$ inducing an H_*-isomorphism by Mayer-Vietoris applied to the obvious cover. It remains to argue about the fundamental group. Applying the Seifert-van Kampen (for fundamental groupoids) to this cover shows that the fundamental groupoids are equivalent. We conclude that $\operatorname{cyl} f \simeq \operatorname{cyl} f'$. ∎

John: the most important thing you should take away from a point-set topology course is that a closed embedding of compact Hausdorff spaces is a cofibration.

Likewise for the homotopy pushout. Given $Y \leftarrow X \to Z$ the homotopy pushout is $Y \sqcup_{X \times 0} X \times [0,1] \sqcup_{X \times 1} Z$. This is homotopy invariant as well, which is proved in an identical fashion.

Recall that Δ is the category of finite nonempty ordered sets with nondecreasing functions between them.

Definition 21. A **simplicial space** is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Top}$. The **geometric realization** $|X_\bullet|$ is the colimit

$$|X_\bullet| \longleftarrow \textstyle\coprod_{n\geq 0} X_n \times \Delta^n \leftleftarrows \coprod_{[m]\to[l]} X_l \times \Delta^m$$

The basic principle is that the "generators" are given by coproducts and the "relations" are given by reflexive coequalizers. For homotopy colimits the generators will still be coproducts, but the relations will be handled by the geometric realization.

Definition 22. For $X_\bullet$ a simplicial space, the n**th latching object** $L_n X_\bullet$ is

$$L_n X_\bullet = \operatorname*{colim}_{(\Delta^{\mathrm{op}}_{<n})_{/[n]}} X_m \subset X_n \tag{1}$$

Think of this as all the degenerate simplices induced from everything below n.

The index category is the category of maps $[n] \to [m]$ for $m < n$.

We say that X is **Reedy cofibrant** if the map $L_n X\bullet \to X_n$ is a cofibration for all n.

Lemma 23. *If we have a map of simplicial spaces $X_\bullet \to Y_\bullet$ such that both X and Y are Reedy cofibrant with the induced maps $X_n \simeq Y_n$ homotopy equivalences then $|X_\bullet| \simeq |Y_\bullet|$.*

Proof outline. We proceed by induction on skeleta. In particular we have the geometric realization of the n-skeleton

$$|\operatorname{sk}_n X_\bullet| \longleftarrow \textstyle\coprod_{k\leqslant n} X_k \times \Delta^k \leftleftarrows \coprod_{[m]\to[l];m,l\leqslant n} X_l \times \Delta^m$$

These skeleta sit inside the total geometric realization as closed embeddings whence $|X_\bullet| = \lim |\operatorname{sk}_n X_\bullet|$. So we will prove $|X_\bullet| \simeq |Y_\bullet|$ by proving that $|\operatorname{sk}_n X_\bullet| \simeq |\operatorname{sk}_n Y_\bullet$. The base case just says that $\operatorname{sk}_0 X_\bullet X_0 \simeq Y_0 = \operatorname{sk}_0 Y_0$. For the inductive step check that there is a pushout

$$\begin{tikzcd}
L_n X \times \Delta^n \coprod_{L_n X \times \partial\Delta^n} X_n \times \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X_\bullet| \arrow[d] \\
X_n \times \Delta^n \arrow[r] & \operatorname{sk}_n X_\bullet|
\end{tikzcd}$$

Likewise for Y. By the inductive hypothesis we know that the map from the top right of the diagram for X to the top right of the diagram for Y is a homotopy equivalence. By assumption the same is true for the bottom left corner. Similarly one has to prove that the top left is a homotopy equivalence. It is then important that the top and left arrows are cofibrations to conclude that the n-skeleton of X is homotopy equivalent to the n-skeleton of Y. $\square$

Write out the details of this proof as homework 4.

Let's now turn to homotopy colimits. Given $\mathcal{C}$ a category we have a simplicial object $N\mathcal{C}_* : \Delta^{\mathrm{op}} \to \mathsf{Set}$, the nerve of $\mathcal{C}$. Observe that the ordinary colimit always receives a surjective map from the coproduct of the functor applied to all the objects in the indexing category. In particular the colimit will always be this coproduct quotiented by a relation coming from morphisms in $\mathcal{C}$. For homotopy colimits we will get a map $F : N\mathcal{C} \to \mathsf{Top}$ sending $[p] \mapsto \sqcup_{N\mathcal{C}_p} F$ and $\operatorname{hocolim}_{\mathcal{C}} F = |F_\bullet|$.

8. Homotopy colimits, continued [10/06/2017]

8.1. **Homotopy colimits.** Recall that if we have an ordinary functor $F : \mathcal{C} \to \mathsf{Top}$ then the colimit $\operatorname{colim}_{\mathcal{C}} F$ can be expressed as a coequalizer: a quotient of the coproduct of $F(c)$ for all $c \in \mathcal{C}$ by the maps in $\mathcal{C}$ (every colimit is a reflexive coequalizer of coproducts).

Definition 24. Given $F : \mathcal{C} \to \mathsf{Top}$ we write $F_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Top}$ for the functor sending $[n] \mapsto \coprod_{N\mathcal{C}_n} F(c_0)$ (where c_0 is the first object in the simplex). The simplicial structure maps are given by copmosition and identities as usual. Then we define the **Bousfield-Kan** homotopy colimit

$$\operatorname{hocolim}_{\mathcal{C}} F := |F_\bullet|$$

Notice that every homotopy colimit is a geometric realization of coproducts.

Theorem 25 (Homotopy invariance of hocolim)**.** *Suppose we have two functors $F, G : \mathcal{C} \to \mathsf{Top}$ such that $F(c)$ and $G(c)$ are cofibrant (i.e. CW complexes) for all $c \in \mathcal{C}$, and there is a natural transformation α such that $\alpha(c)$ is a homotopy equivalence. Then* $\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{C}} G$.

Proof. This is homework 4 (from last time). Recall that the lemma from last time tells us that given a map of Reedy cofibrant simplicial spaces $X_\bullet \to Y_\bullet$ inducing equivalences on n-simplices for every n, the geometric realizations are equivalent. Hence we need only check Reedy cofibrancy for $F_\bullet$ and $G_\bullet$.

In this case the nth latching object of $F_\bullet$ is

$$L_n F_\bullet = \coprod F(c_0)$$

where the coproduct is taken over all degenerate n-simplices of $N\mathcal{C}$. But by the CW complex assumption above the maps $L_n F_\bullet \hookrightarrow F_\bullet$ is a cofibration, as desired. □

8.2. **Factorization homology—a predefinition.**

Definition 26. We define $\mathsf{Disk}_{n/M}$ to be the category of n-disks embedding in M with morphisms given by inclusion (it is equivalent to the subposet of opens on M such that the image is diffeomorphic to an n-disk).

We can make the following predefinition (easier to make, harder to work with). Given $A : \mathsf{Disk}_n \to \mathsf{Top}$ we define the factorization homology

$$\int_M A := \operatorname{hocolim}_{\mathsf{Disk}_{n/M}} A.$$

Really we should be working with the topological version $\mathscr{D}\mathsf{isk}_{n/M}$ but it will end up being homotopy equivalent.

We want factorization homology $\int_M A$ to be M, where we replace $\mathbb{R}^n$ with $A(\mathbb{R}^n)$.

Example 27 (Desiderata)**.**

(1) if $A = *$? Then we would like $\int_M * \simeq *$.
(2) if $A(\coprod_I \mathbb{R}^n) = \coprod_I \mathbb{R}^n$ then $\int_M \mathrm{id} \simeq M$.
(3) be able to compute $\int_M A$ for A belonging to the examples we discussed earlier. For instance, commutative algebras, n-fold loop spaces, free n-disk algebras, trivial n-disk algebras, and enveloping algebra of a Lie algebra.
(4) if A lands in Ch sending $\coprod_I \mathbb{R}^n \mapsto A^{\oplus I}$ then $\int_M A \simeq C_*(M, A)$ (and likewise for spectra).

We need tools for computing homotopy colimits. For instance, it is useful to introduce the topological version,

$$\mathsf{Disk}_{n/M} \to \mathscr{D}\mathsf{isk}_{n/M},$$

and it turns out that homotopy colimits over these two categories are equivalent. To make statements like this, we need crieria for when two homotopy colimits are equivalent when they're indexed by different categories.

Since we don't have time left today to introduce ∞-categories, let's go over some properties of hocolim.

Theorem 28 (Quillen's theorem A)**.** *Let $g : \mathcal{C} \to \mathcal{D}$ is a functor. If F is some functor from $\mathcal{D}$ to some target (such as topological spaces). Then*

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{D}} F$$

if and only if g is ***final****. In other words, for $d \in \mathcal{D}$, define $\mathcal{C}^{d/} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{d/}$, and say that g is final if $B(\mathcal{C}^{d/}) \simeq *$ where $B\mathcal{C} := \operatorname{hocolim} *$.*

There is another key property of homotopy colimits involving hypercovers. Suppose we have a functor $\mathcal{C} \to \mathrm{Opens}(X) \hookrightarrow \mathsf{Top}$. When is

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq X?$$

Define, for $x \in X$, $\mathcal{C}_x$ to be the full subcategory of objects c such that $x \in F(c)$. If $B\mathcal{C}_x \simeq *$ for each $x \in X$ then $\operatorname{hocolim}_{\mathcal{C}} F \simeq X$.

Exercise 29. $\operatorname{hocolim}_* F = F(*)$.

9. ∞-CATEGORIES [10/09/2017]

9.1. **Topological enrichment.** Suppose we have a category $\mathcal{T}$ with products as well as a functor $\Delta \to \mathcal{T}$ from the ordinal category. Then $\mathrm{Maps}_{\mathcal{T}}(s,t)$ is a simplicial set with

$$\mathrm{Maps}_{\mathcal{T}}(s,t)_p = \mathrm{Hom}_{\mathcal{T}}(s \times [p], t),$$

where by $[p]$ we denote the image of the functor. If we now apply geometric realization, we obtain mapping spaces.

Consider for example $\mathcal{T} = \mathsf{Top}$ (as a non-enriched, ordinary category). There is a functor $\Delta \to \mathsf{Top}$ which sends $[p]$ to the geometric p-simplex. Then we get a simplicial set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)_\bullet$, and notice that

$$\mathrm{Maps}_{\mathsf{Top}}(X \times \Delta^p, Y) \cong \mathrm{Maps}_{\mathsf{Top}}(\Delta^p, \mathrm{Maps}_{\mathsf{Top}}(X,Y))$$

where we equip the set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)$ with the compact-open topology, as usual. Hence the simplicial set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)_\bullet$ is isomorphic to the singular simplicial set $\mathrm{Sing}\,\mathrm{Maps}_{\mathsf{Top}}(X,Y)$. If we apply the geometric realization, since $|\mathrm{Sing}\,A| \simeq A$, we see that we obtain the usual topological enrichment (at least up to homotopy) on the category Top.

This trick allows us to enrich various categories in Top. As we have seen above the category of topological spaces is an immediate example, and it is not hard to do similarly for the category of simplicial sets. Another two familiar examples are those of chain complexes and natural transformations of functors. For a slightly unfamiliar example one could use the functor $\Delta \to \mathsf{CAlg}_{\mathbb{R}}^{\mathrm{op}}$ of de Rham forms on simplices, which sends $[p] \mapsto \Omega^*(\Delta^p)$, to give the (opposite) category of commutative $\mathbb{R}$-algebras a topological enrichment.

This leads us to the following general idea, which highlights the importance of topological enrichment.

> **Principle:** Everywhere where there is a notion of homotopy, there exists an enrichment in Top such that this is an actual homotopy.

9.2. **Complete Segal spaces and quasicategories.** Now, whatever ∞-categories are, they should have two properties:

1. The collection of ∞-categories up to some notion of equivalence should be equal to the collection of topological categories modulo homotopy equivalence (see below for the formal definition).
2. Colimits, limits, functor categories, over/undercategories in ∞-categories are homotopy colimits, homotopy limits, etc. in the corresponding topological category.

Definition 30. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between topological categories. We say that F is a **homotopy equivalence** if for each $c, c' \in \mathcal{C}$, $\mathrm{Maps}_{\mathcal{C}}(c,c') \simeq \mathrm{Maps}_{\mathcal{D}}(Fc, Fc')$ and every object $d \in \mathcal{D}$ is homotopy equivalent to some $F(c)$ for $c \in \mathcal{C}$. In other words, there exists a map $d \to Fc$ such that $\mathrm{Maps}_{\mathcal{D}}(e,d) \to \mathrm{Maps}(e, Fc)$ is a homotopy equivalence for all $e \in \mathcal{D}$.

So that's roughly the philosophy of ∞-categories. They are a nice ground to work on when dealing with homotopy invariance. When it comes to actually defining ∞-categories, there is a conceptual option and a more economical option: complete Segal spaces and quasicategories, respectively. Let me tell you briefly about complete Segal spaces.

When we are given $\mathcal{C}$ a category, there is a set of objects and a set of morphisms. However, the only way we ever use categories is up to equivalence, and these underlying sets have no invariance properties with respect to equivalence of categories (for instance the sets of objects or corresponding sets of morphisms need not have the same number of elements). This leads us to the question: how can we think of a category in a way that better reflects the homotopy theory (i.e. equivalences) of categories.

It turns out that we can construct *spaces* of objects and morphisms of $\mathcal{C}$ in the following way. Consider the underlying groupoid $\mathcal{C}^0 \subset \mathcal{C}$ where we have thrown out all the noninvertible maps. Taking the nerve (classifying space) $N\mathcal{C}^0$ gives us a simplicial set. The associated space is of course the geometric realization $|N\mathcal{C}^0|$. For morphisms, consider the category $\mathrm{Fun}^{\mathrm{iso}}([1], \mathcal{C})$ of functors $[1] \to \mathcal{C}$ with natural transformations through isomorphisms. This category is a also a groupoid, so we obtain a space $|N\,\mathrm{Fun}^{\mathrm{iso}}([1], \mathcal{C})|$.

Observe now that if we have two equivalent categories $\mathcal{C} \simeq \mathcal{C}'$ then the spaces of objects and morphisms that we have defined above will be homotopy equivalent. Generalizing these constructions for higher $[p]$ we obtain a fully faithful functor

$$C_\bullet : \mathsf{Cat} \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Top})$$

sending a category $\mathcal{C}$ to the simplicial space that sends $[p] \mapsto N\,\mathrm{Fun}^{\mathrm{iso}}([p], \mathcal{C})$. It moreover has the property that the diagram

$$\begin{tikzcd} \mathcal{C}_\bullet[2] \arrow[r] \arrow[d] & \mathcal{C}_\bullet\{1<2\} \arrow[d] \\ \mathcal{C}_\bullet\{0<1\} \arrow[r] & \mathcal{C}_\bullet\{1\} \end{tikzcd}$$

is actually a homotopy pullback square (this turns out to more or less characterizes the image of $C_\bullet$). In particular, one should suspect (correctly) that colimits and limits will be mapped to homotopy colimits and homotopy limits. Unfortunately, going down this path to ∞-categories quickly turns into messing around with bisimplicial sets, which starts to get a bit complicated.

This leads us to the more economical option of quasicategories. For quasicategories there is only one simplicial index involved and there is the important advantage that there are thousands of pages of reference material.

Definition 31. A **quasicategory** $\mathcal{C}$ is a simplicial set such that every inner horn (for $n \geq 2$) has a filler.

Let's explain these terms. Write $\Delta[n]$ for the geometric n-simplex (the functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ given $\Delta[n] = \mathrm{Hom}_\Delta(-, [n])$). There are a number of maps $\Delta[n-1] \to \Delta[n]$ induced by maps $[n-1] \to [n]$ that skipping some i. Then the ith horn of the geometric n-simplex is defined to be

$$\Lambda_i[n] = \bigcup \Delta[n-1]$$

where the union is over all faces $\Delta[n-1] \hookrightarrow \Delta[n]$ except for the ith. For instance there are three horns of the 2-simplex. An inner horn is a horn where the missing face is neither the 0th face or the nth face (so the 2-simplex has only one inner horn, $\Lambda_1[2]$). Now we can define what it means for a simplicial set $\mathcal{C}$ to have inner horn fillings. It means that for every map $\Lambda_i[n] \to \mathcal{C}$ there is a lift, or "filling",

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

of the map to the simplex making the diagram commute.

Example 32. Spaces and categories are two natural sources of quasicategories.

(1) Consider $\mathcal{C} = \mathrm{Sing}(X)$. By the adjunction between geometric realization and Sing the data of a map $\Lambda_i[n] \to \mathrm{Sing}\, X$ is precisely the data of a map $|\Lambda_i[n]| \to X$. Now one can choose (there is no unique choice) say a retraction $|\Delta[n]| \to |\Lambda_i[n]|$. Composing with the map to X and again applying adjointness, we obtain a map $\Delta[n] \to \mathrm{Sing}\, X$ making the relevant diagram commute. We conclude that the singular simplicial set of a space is a quasicategory.

(2) Given a category C consider the nerve $\mathcal{C} = NC$. By the Yoneda lemma a map $\Lambda_1[2] \to NC$ is precisely the data of a composable pair of morphisms in C. In particular there is a unique way of filling the horn into the simplex by using the composition of these two maps. A similar argument holds for higher-dimensional horns. We conclude that the nerve of any category is a quasicategory.

What we will do next is give the definitions of colimits, limits, functor categories, and over/undercategories in quasicategories. We will also discuss a variant of the nerve functor, $N : \mathsf{TopCat} \to \mathsf{QuasiCat}$, which in good cases will send hocolim $\mapsto$ colim.

Grisha: I understand your philosophy that complete Segal spaces are more compelling than quasicategories. Is there some concrete statement that backs this claim up? **John**: I don't know if this is getting at your question but here's an example. There is an inclusion $\mathsf{Cat}_\infty \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Top})$ so it's easy to give an internal definition of the ∞-category of ∞-categories. However the corresponding construction is not so easy with quasicategories.

10. Colimits in ∞-categories [10/11/2017]

10.1. **Colimits in 1-categories.** Recall the definition of a colimit of a functor $F : \mathcal{C} \to \mathcal{D}$ in the theory of ordinary categories. We define the **right cone** $\mathcal{C}^{\triangleright}$ of $\mathcal{C}$ as follows. It has objects the objects of $\mathcal{C}$ together with an object we denote $*$. For morphisms we take

$$\mathrm{Hom}_{\mathcal{C}^{\triangleright}}(x, y) = \begin{cases} * & y = * \\ \varnothing & x = * \\ \mathrm{Hom}_{\mathcal{C}}(x, y) & \text{otherwise} \end{cases}$$

Next we define the **undercategory** $\mathcal{D}^{F/}$ as the fiber

$$\begin{tikzcd} \mathcal{D}^{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) \arrow[d] \\ \{F\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{tikzcd}$$

What are the morphisms in the undercategory?

It has as objects pairs $d \in \mathcal{D}$ with a natural transformation $F \implies \underline{d}$, where $\underline{d}$ is the constant functor. With these definitions we can now define colimits.

Definition 33. An object $d \in \mathcal{D}$ is a **colimit** of the functor $F : \mathcal{C} \to \mathcal{D}$ if there exists a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ with $\overline{F}(*) \cong d$ and such that the natural restriction $\mathcal{D}^{\overline{F}/} \to \mathcal{D}^{F/}$ is an equivalence.

If you're a bit confused, like Nilay is, about why this is a colimit, observe that the category $\mathcal{C}^{\triangleright}$ has a final object. For any $\mathcal{C}'$ which has a final object, if we have a functor $G : \mathcal{C}' \to \mathcal{D}$ then there is an equivalence $\mathcal{D}^{G/} \cong \mathcal{D}^{G'/}$, where $G' : \mathcal{C} \to \mathcal{D}$ is the restriction of G.

10.2. **Colimits in quasicategories.** We'd like to make a similar definition for quasicategories. We will need to be able to define equivalence, right cones, and undercategories in that context.

Definition 34. For $\mathcal{C}$ a quasicategory and any objects x and y (i.e. $x, y \in \mathcal{C}[0]$) we define the **mapping space** $\mathrm{Maps}_{\mathcal{C}}(x, y)$ as the fiber

$$\begin{tikzcd} \mathrm{Maps}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathrm{Maps}(\Delta[1], \mathcal{C}) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\ \{x, y\} \arrow[r] & \mathrm{Maps}(\Delta[0], \mathcal{C}) \times \mathrm{Maps}(\Delta[0], \mathcal{C}) \end{tikzcd}$$

Here $\mathrm{Maps}(X, Y)$ for X and Y simplicial sets is a simplicial set whose set of n-simplices is the set $\mathrm{Hom}_{\mathsf{sSet}}(X \times \Delta[n], Y)$, i.e. the internal hom. The mapping space is a priori just a simplicial set.

Recall that Kan complexes—simplicial sets that satisfy the horn filling condition for all horns (not just inner horns)—are the combinatorial analog of spaces.

Lemma 35. *As defined above,* $\mathrm{Maps}_{\mathcal{C}}(x, y)$ *is a Kan complex.*

Definition 36. If $F : \mathcal{C} \to \mathcal{D}$ is a functor between quasicategories (i.e. a map of simplicial sets), then F is a **categorical equivalence** if

(1) the induced map $F : h\mathcal{C} \to h\mathcal{D}$ is an equivalence of categories, where $h\mathcal{C}$ is the category with objects that of $\mathcal{C}$ and morphisms the set $\pi_0 \mathrm{Maps}(X, Y)_n$.

(2) for any $x, y \in \mathcal{C}$ the induced map $F : \mathrm{Maps}_{\mathcal{C}}(x, y) \to \mathrm{Maps}_{\mathcal{D}}(Fx, Fy)$ is a homotopy equivalence (equivalently a homotopy equivalence after geometric realization).

Now we need the notion of the right cone of a simplicial set. Well for a hint of what this definition should be, let's look at the nerve of the right cone construction above:

$$\begin{aligned} N(\mathcal{C}^{\triangleright})_k = \mathrm{Fun}([k], \mathcal{C}^{\triangleright}) &= * \sqcup \coprod_{i=0}^{k} \mathrm{Fun}([i], \mathcal{C}) \\ &= * \sqcup \coprod_{i=0}^{k} N(\mathcal{C})_i. \end{aligned}$$

This leads us to the following definition.

Definition 37. For S a simplicial set, define the **right cone** on S to be

$$S_k^{\triangleright} = * \sqcup \coprod_{i \leqslant k} S_i.$$

One checks that this naturally forms a simplicial set.

Definition 38. For $F : \mathcal{C} \to \mathcal{D}$ a functor of quasicategories, we define the **undercategory** $\mathcal{D}^{F/}$ to be the fiber

$$\begin{tikzcd} \mathcal{D}^{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) \arrow[d] \\ \{F\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{tikzcd}$$

Definition 39. We say that $d \in \mathcal{D}$ is a **colimit** of F if there exists a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ with $\overline{F}(*) = d$ and $\mathcal{D}^{F/} \simeq \mathcal{D}^{\overline{F}/}$ is a categorical equivalence.

Theorem 40.

(1) Colimits in a quasicategory are invariant upto categorical equivalence. In other words, if we have an equivalence $\mathcal{C} \simeq \mathcal{C}'$ and $\mathcal{D} \simeq \mathcal{D}'$ with a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r] & \mathcal{D}' \end{tikzcd}$$

then $\mathrm{colim}_{\mathcal{C}} F = \mathrm{colim}_{\mathcal{C}'} F'$. What does this mean?

(2) The simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is a quasicategory if* $\mathcal{C}, \mathcal{D}$ *are quasicategories, and is invariant up to categorical equivalence. In other words, there is a categorical equivalence of quasicategories* $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}', \mathcal{D}')$.

The proof is a straightforward exercise in model categorical language and we might work through this in the future. First let's explain why these results are so great, and why they motivate working with quasicategories.

Consider $\mathscr{D}\mathsf{isk}_{\infty}^{\mathrm{rect}} = \varinjlim \mathscr{D}\mathsf{isk}_{n}^{\mathrm{rect}}$ the sequential limit. One finds that $\mathscr{D}\mathsf{isk}_{\infty}^{\mathrm{rect}} \simeq \mathsf{Fin}$. For motivation for why this might be true, recall that $\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \simeq$

S^{n-1} and as n grows large we obtain S^∞, which is contractible. A basic question one might ask is whether there is a factorization

$$\begin{tikzcd} \mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}} \arrow[r] \arrow[d] & \mathsf{Top} \\ \mathsf{Fin} \arrow[ur, dashed] & \end{tikzcd}$$

It turns out that there does not exist such a factorization in general:

$$\mathrm{Fun}(\mathsf{Fin}, \mathsf{Top}) \not\simeq \mathrm{Fun}(\mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}}, \mathsf{Top})$$

This is stemming from the fact that infinite loop spaces are not equivalent to topological groups.

Expand on what this has to do with ∞-categories. Is the point that passing to ∞-categories will yield an equivalence?

11. Homotopy invariance I [10/13/2017]

Recall there was a homework problem to show that there is a continuous assignment $\mathrm{Maps}_c(U,Z) \to \mathrm{Maps}_c(V,Z)$. I should have specified that we are to give $\mathrm{Maps}_c(U,Z)$ the subspace topology as inherited from $\mathrm{Maps}_*(U^+,Z)$. If we view it as a subspace of $\mathrm{Maps}(U,Z)$ this is statement is not true. Of course, this did not seem to prevent you from proving it. . . you know what they say—where there's a will, there's a way. Anyway, for the next homework revise that solution. In addition, do the following for **homework.**

Exercise 41. Prove that there is a homeomorphism $|\Delta[n]| \cong \Delta^n$. Moreover, show that the geometric realization $|X|$ of a simplicial set X has the structure of a CW complex with an n-cell for each nondegenerate n-simplex.

I want to give you a good taste of proofs in quasicategory theory, without having to prove absolutely everything. The following (the homotopy invariance of colimits) should be a good pedagogical example with which we can "get in and get out" of the theory of quasicategories. The main reference will of course be Jacob Lurie's Higher Topos Theory [Lur09].

Proposition 42 (HTT proposition 1.2.9.3)**.** *Let $p : \mathcal{C} \to \mathcal{D}$ be a map of quasicategories and $j : K \to \mathcal{C}$ be any map. Then if p is an equivalence, so is the induced map*

$$\mathcal{C}^{j/} \xrightarrow{\sim} \mathcal{D}^{p\circ j/}.$$

Matt: how does this relate to the notion of pointwise homotopy invariance? John: this result is a bit harder than that one.

How does this imply the statement last lecture about colimits?

Let's outline the proof:

(1) $\mathcal{C}^{j/}$ is a quasicategory
(2) $\mathcal{C}^{j/} \to \mathcal{C}$ is a left fibration
(3) Given two left fibrations $\mathcal{C}', \mathcal{C}''$ over $\mathcal{C}$, and a compatible map g between them, then g is an equivalence if it is an equivalence on fibers. In other words it is an equivalence if for all $x \in \mathcal{C}$ the map $\mathcal{C}' \times_{\mathcal{C}} \{x\} \to \mathcal{C}'' \times_{\mathcal{C}} \{x\}$ is an equivalence of Kan complexes.

We'll begin by showing (2). We first need a definition.

Definition 43. For X, Y simplicial sets, the **join** $X \star Y$ is the simplicial set given on totally ordered sets as

$$(X \star Y)(J) = \coprod_{J = I \coprod I'} X(I) \times Y(I')$$

where in the coproduct, every element of I is less than every element of I'. Equivalently,

$$(X \star Y)([n]) = X_n \sqcup \left(\coprod_{i+j=n-1} X_i \times Y_j \right) \sqcup Y_n$$

In the case where C, D are categories, one can check that the usual join $C \star D$, which has

$$\mathrm{Hom}_{C\star D}(x,y) = \begin{cases} \mathrm{Hom}(x,y) & x, y \in C \text{ or } x, y \in D \\ * & x \in C, y \in D \\ \varnothing & \text{otherwise} \end{cases}$$

has the property that its nerve is the quasicategorical join of the corresponding nerves of C and D. A similar statement is true for spaces. Another thing to check is that $X^{\triangleright} = X \star \Delta[0]$. **Homework 6:** Check that $\Delta[n] \star \Delta[m] \cong \Delta[n+m+1]$

Definition 44. A class of morphisms $S \subset C$ (for C an ordinary category) is **weakly saturated** if it is

(1) closed under pushouts,
(2) closed under (transfinite) composition,
(3) and closed under retracts.

This notion is important because any map that has a lifting property with respect to some class of morphisms S then it will also have the lifting property with respect to the weakly saturated closure.

Did I say this correctly?

Definition 45. We say that $A \to B$ is **left/right/inner anodyne** if it belongs to the smallest weakly saturated class containing (for $n \geq 1$), $\{\Lambda_i[n] \hookrightarrow \Delta[n], i < n\}$ (left), $\{\Lambda_i[n] \hookrightarrow \Delta[n], i > n\}$ (right), $\{\Lambda_i[n] \hookrightarrow \Delta[n], 0 < i < n\}$ (inner).

Notice that "anodyne" is an english word meaning inoffensive, bland, or unproblematic.

Lemma 46 (HTT proposition 2.1.2.3)**.** *Given inclusions of simplicial sets $f : A_0 \subset A, g : B_0 \subset B$ such that f is right anodyne or g is left anodyne, then*

$$A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \hookrightarrow A \star B$$

is inner anodyne.

Proof. The two cases are dual so we will just do the case where f is right anodyne. Consider the class of all morphisms $f : A_0 \to A$ for which the inclusion of the proposition is inner anodyne. This class is weakly saturated whence it suffices to show that it contains $\Lambda_i[n] \subset \Delta[n]$ for $0 < i \leqslant n$. We thus suppose that f is of this form. Similarly for g: reduce $g : \partial\Delta[m] \subset \Delta[m]$. We now have

$$\Lambda_i[n] \star \Delta[m] \coprod_{\Lambda_i[n] \star \partial\Delta[m]} \Delta[n] \star \partial\Delta[m] \hookrightarrow \Delta[n+m+1].$$

More homework: $\Lambda_i[n] \star \partial\Delta[m] \cong \Lambda_i[n+m+1]$, which concludes the proof. ☐

Definition 47. We say that $X \to Y$ is a **inner/left/right fibration** if it has the right lifting property with respect to inner/left/right anodyne maps.

Proposition 48 (HTT proposition 2.1.2.1)**.** *Given $A \subset B \xrightarrow{p} X \xrightarrow{q} S$, with $r = q \circ p$ and $r_0 : A \subset B$, with q an inner fibration. Then $X^{p/} \to X^{p_0/} \times_{S^{r_0/}} S^{r/}$ is a left fibration.*

This is all to show that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration (and that the domain is a quasicategory).

Sam: what does a left fibration geometrically realize to? A quasifibration? John: Yeah. [Correction next lecture: I meant to say no. There is a paper of Quillen in the Annals, titled something like: "the geometric realization of a Kan fibration is a Serre fibration." You can guess what the main theorem is. It turns out that for $X \to Y$ a left (or right) fibration it is not necessarily true that $|X| \to |Y|$ is a quasifibration. As an example, consider the left fibration $\mathcal{C}^{x/} \to \mathcal{C}$. Consider the fiber

$$\begin{tikzcd} \mathrm{Maps}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}^{x/} \arrow[d] \\ \{y\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

This cannot possibly yield a quasifibration. Consider a y' with a map $y \to y'$. We get a similar fiber $\mathrm{Maps}_{\mathcal{C}}(x,y)$. But there is no reason for these mapping spaces to be homotopy equivalent (and similarly after taking geometric realization).

Let's recall Quillen's theorem B. Given $\mathcal{C} \to \mathcal{D}$ and the fiber diagram

$$\begin{tikzcd} \mathcal{C}^{d/} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \{d\} \arrow[r] & \mathcal{D} \end{tikzcd}$$

one might ask when taking classifying spaces yields again a homotopy pullback. This is true if for all $d \to c$ in $\mathcal{D}$ we have that $B\mathcal{C}^{d/} \simeq B\mathcal{C}^{c/}$.]

12. Homotopy invariance II [10/16/2017]

Today we'll discuss the proof of the following fact, which was (1) in our outline proof of homotopy invariance.

Corollary 49 (HTT 2.1.2.2)**.** *For all $K \xrightarrow{p} \mathcal{C}$ the associated undercategory $\mathcal{C}^{p/}$ is a quasicategory.*

Proof. We claim that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration. Left implies inner, so composing with the map to the point implies that $\mathcal{C}^{p/} \to \mathcal{C} \to \Delta[0]$ is an inner fibration (it is easy to check that compositions of fibrations are fibrations directly from the lifting property). We conclude that $\mathcal{C}^{p/}$ is a quasicategory.

It remains to show that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration, which we do below. □

Recall from last time we had shown (if you include the homework) the following.

Lemma 50 (HTT 2.1.2.3)**.** *Given $f : A_0 \hookrightarrow A, g : B_0 \hookrightarrow B$ with either f right anodyne or g left anodyne then*

$$A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \hookrightarrow A \star B$$

is inner anodyne.

This immediately implies

Proposition 51 (HTT 2.1)**.** *Given $A \subset B \xrightarrow{p} X \xrightarrow{q} S$ with the inclusion denoted r_0 and the composition $q \circ p =: r$ where q is an inner fibration, then*

$$X^{p/} \to X^{p_0/} \times_{S^{r_0/}} S^{r/}$$

is a left fibration.

Proof. Recall that the data of a map $J \to \mathcal{C}^{p/}$ for $p : K \to \mathcal{C}$ is precisely the data of a map $K \star J \to \mathcal{C}$ such that $K \star \varnothing \to \mathcal{C}$ is p. To check that the given map is a left fibration we look at a diagram

$$\begin{tikzcd}
\Lambda_k[n] \arrow[r] \arrow[d] & X^{p/} \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & X^{p_0/} \times_{S^{r_0/}} S^{r/}
\end{tikzcd}$$

Let's apply our adjunction to obtain compatible maps

$$B \star \Lambda_k[n] \longrightarrow X$$

$$A \star \Delta[n] \longrightarrow X$$

$$B \star \Delta[n] \longrightarrow S$$

$$A \star \Delta[n] \longrightarrow S$$

Putting these together,

$$\begin{tikzcd} B \star \Lambda_k[n] \cup_{A \star \Lambda_k[n]} A \star \Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d] \\ B \star \Delta[n] \arrow[r] & S \end{tikzcd}$$

and applying the lemma above, the vertical map on the left is inner anodyne, whence because $X \to S$ is an inner fibration, there exists a lift. ∎

This concludes the proof that the undercategory is a quasicategory. To see this, we apply the proposition to the case where $X = \mathcal{C}$, $A = \varnothing$, and $B = *$. Hence $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration. Huh?

Sean: is there a time when it matters that these were left fibrations and not just inner? John: absolutely. Think of inner as a technical condition but left/right as a homotopy invariant property. In particular, every functor is equivalent to an inner fibration. This is not at all true for left fibrations. In particular, $\mathsf{LFib}_{\mathcal{D}} \simeq \mathrm{Fun}(\mathcal{D}, \mathsf{Spaces})$—they're like "fiber bundles with connection on $\mathcal{D}$."

Proposition 52 (HTT 1.2.5.1)**.** *For $\mathcal{C}$ a simplicial set the following are equivalent:*

(1) $\mathcal{C}$ is a quasicategory and $\mathsf{h}\mathcal{C}$ is a groupoid;
(2) $\mathcal{C} \to *$ is a left fibration;
(3) $\mathcal{C} \to *$ is a right fibration;
(4) $\mathcal{C} \to *$ is a Kan fibration, i.e. $\mathcal{C}$ is a Kan complex.

*If any of these are try we call $\mathcal{C}$ an **∞-groupoid** or **space**.*

Proposition 53 (HTT 1.2.4.3)**.** *A morphism $\phi : \Delta[1] \to \mathcal{C}$ in a quasicategory $\mathcal{C}$ is an equivalence if and only if for any extension $f_0 : \Lambda_0[n] \to \mathcal{C}$*

$$\begin{tikzcd} \Delta[1] \arrow[r, "\phi"] \arrow[d, hook] & \mathcal{C} \\ \Lambda_0[n] \arrow[ur] \arrow[d, hook] & \\ \Delta[n] \arrow[uur, dashed] & \end{tikzcd}$$

there is a lift to a map $f : \Delta[n] \to \mathcal{C}$.

Proof. By adjunction

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C}/\Delta[n-2] \arrow[d] \\ \Delta[0<1] \arrow[r] \arrow[ur, dashed] & \mathcal{C}/\partial\Delta[n-2] \end{tikzcd}$$

That proves one direction. For the other direction take a map $\phi : x \to y$. We have a filler $\Lambda_0[2] \hookrightarrow \Delta[2] \xrightarrow{\psi} \mathcal{C}$ call it σ. This 2-simplex σ gives a homotopy $\mathrm{id}_x \simeq \psi \circ \phi$. Show that $\phi \circ \psi \simeq \mathrm{id}_y$ for **homework.** ∎

This implies the equivalence of (1) $\iff$ (2) and dually (1) $\iff$ (2). But then (1) $\iff$ (2) + (3) = (4).

13. Homotopy invariance III [10/18/2017]

Recall last time we wanted to prove Proposition 1.2.5.1.

Proof. To show that (1) $\implies$ (2) notice that every $\Delta[1] \to \mathcal{C}$ is a homotopy equivalence. Choose any f_0

$$\begin{tikzcd} \Lambda[n] \arrow[r, "f_0"] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

and now by the previous proposition there exists an extension

$$\begin{tikzcd} \Delta[0<1] \arrow[d] \arrow[dr] & \\ \Lambda_0[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

To see that (2) $\implies$ (1) draw the same picture and apply the proposition, which implies that ϕ is an equivalence.

Notice that by taking opposites (1) $\iff$ (2) implies (1) $\iff$ (3), since taking opposites takes left fibrations to right fibrations. Hence (1) $\iff$ (2) + (3). But being a left and right fibration is the same as being a Kan fibration, which completes the proof. $\square$

Figure out the HTT number. Fix the notation to tilde.

Corollary 54. *If $\mathcal{C}$ is a quasicategory there exists a maximal sub-Kan complex $\mathcal{C}^0$ whose morphisms consist of the homotopy equivalences in $\mathcal{C}$.*

Proof. We can define $\mathcal{C}^0$ as the subsimplicial set with 1-simplices the homotopy equivalences. $\mathcal{C}^0$ is a quasicategory, and $\mathsf{h}\mathcal{C}^0$ is a groupoid if and only if $\mathcal{C}^0$ is a Kan complex. $\square$

In particular we have $\mathsf{Kan} \subset \mathsf{QCat} \subset \mathsf{sSet}$ and the construction in the corollary is the right adjoint to the inclusion $\mathsf{Kan} \hookrightarrow \mathsf{QCat}$.

Recall that our purpose was to show that colimits in quasicategories are invariant with respect to categorical equivalence. In particular, given $J \xrightarrow{j} \mathcal{C} \xrightarrow{p} \mathcal{D}$ where p is a categorical equivalence, we want to show that $\mathcal{C}^{j/} \simeq \mathcal{D}^{p\circ j/}$. We first needed the undercategories to be quasicategories. This we showed last time. Next we show that the two horizontal arrows

$$\begin{tikzcd} \mathcal{C}^{j/} \arrow[r] \arrow[dr] & \mathcal{D}^{p\circ j/} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}^{p\circ j/} \\ & \mathcal{C} & \end{tikzcd}$$

are equivalences. To see that the first is an equivalence we observe first that the vertical maps in the triangle are left fibrations, whence it is enough to show that it produces an equivalence on fibers. Thus we need the following. Given

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "p"] \arrow[d, "g"] & \mathcal{C}'' \arrow[dl] \\ \mathcal{D} & \end{tikzcd}$$

where g and h are left fibrations, we wish to show that p is an equivalence if and only if $\mathcal{C}'_d \to \mathcal{C}''_d$ is an equivalence of Kan complexes for all $d \in \mathcal{D}$. To prove this it's easiest to prove a slightly more general result. Then we need

Lemma 55 (HTT 2.5.4.1)**.** *Given $J \to \mathcal{C} \to \mathcal{D}$ with the map from $\mathcal{C}$ to $\mathcal{D}$ an equivalence, then $\mathcal{C}^{j/} \times_{\mathcal{C}} \{x\} \to \mathcal{D}^{p \circ j/} \times_{\mathcal{D}} \{px\}$ is an equivalence of Kan complexes for all $x \in \mathcal{C}$.*

The following picture is good to keep in mind:

$$\begin{tikzcd} & \text{Kan fibration} \arrow[dl] \arrow[dr] & \\ \text{left fibration} \arrow[d] & & \text{right fibration} \arrow[d] \\ \text{coCartesian fibration} \arrow[dr] & & \text{Cartesian fibration} \arrow[dl] \\ & \text{inner fibration} & \end{tikzcd}$$

Recall that an inner fibration is more of a technical condition rather than having some homotopy invariant meaning. Each of these fibrations (except for inner fibrations) are classified by functors to a representing object.

Example 56. Consider a functor $F : [1] \to \mathsf{Cat}$. Call $F(0) = \mathcal{C}, F(1) = \mathcal{D}$. From this we can construct a category $\mathcal{M} = \mathrm{cyl}(F) := \mathcal{C} \times [1] \coprod_{\mathcal{C} \times \{1\}} \mathcal{D}$ which sits over $[1]$, $\mathcal{M} \to [1]$. This cylinder construction is a map $\mathrm{Fun}([1], \mathsf{Cat}) \to \mathsf{Cat}_{/[1]}$. You should think of this as the most basic example of an "unstraightening construction". The categories you obtain are the coCartesian fibrations over $[1]$.

Definition 57. We say that a **correspondence** between two categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{M} \to [1]$ with $\mathcal{M}_0 \cong \mathcal{C}$ and $\mathcal{M}_1 \cong \mathcal{D}$.

This construction gives us correspondences but not all correspondences arise this way. If we consider a span $\mathcal{E}$ from $\mathcal{C}$ to $\mathcal{D}$ we can produce a correspondence. In particular take the parameterized join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D} = \mathcal{C} \coprod_{\mathcal{E} \times \{0\}} \mathcal{E} \times [1] \coprod_{\mathcal{E} \times \{1\}} \mathcal{D}$ over $[1]$.

If our fibrations in our diagram above are over $\mathcal{C}$ (except for the inner fibration, say), then we get a diagram before unstraightening

$$\begin{tikzcd} & \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}^0_\infty) \arrow[dl] \arrow[dr] & \\ \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}_\infty) \arrow[d] & & \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}^{\mathrm{op}}_\infty) \arrow[d] \\ \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) & & \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}^{\mathrm{op}}_\infty) \end{tikzcd}$$

Note: unstraightening is also known as the "Grothendieck construction." In particular given $F : \mathcal{C} \to \mathsf{Cat}_\infty$ then the fiber is just $F(x)$ over x:

$$\begin{tikzcd} F(x) \arrow[r] \arrow[d] & \mathcal{C}_F \arrow[d] \\ \{x\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

Corollary 58. *Suppose we have a map $\mathcal{C}' \to \mathcal{C}''$ of left fibrations over $\mathcal{C}$, if $\mathcal{C}' = \mathcal{C}_F$ and $\mathcal{C}'' = \mathcal{C}_G$ (unstraightening) with the map of fibrations being induced by a natural transformation α sending $F \implies G : \mathcal{C} \to \mathsf{Gpd}_\infty$. Then the map is an equivalence if and only if α is an equivalence if and only if $F(x) \xrightarrow{\alpha} G(x)$ is an equivalence i.e. $\mathcal{C}'_x \to \mathcal{C}''_x$ is an equivalence.*

We would have to prove a lot of this stuff to prove our fact, so we will probably take it for granted.

Principle: any construction from category theory that only uses universal properties goes through for ∞-categories.

The reason this unstraightening stuff is important because it's generally easier to think about functors out of things instead of fibrations. On the other hand, fibrations (unstraightened things) are generally easier to construct.

14. Back to topology [10/20/2017]

14.1. Finishing up homotopy invariance.

Lemma 59 (HTT 2.4.5.1). *Given $K \xrightarrow{j} \mathcal{C} \xrightarrow{p} \mathcal{D}$ where p is an equivalence, we wish to show that*

$$\mathcal{C}^{j/} \times_{\mathcal{C}} \{x\} \simeq \mathcal{D}^{p \circ j} \times_{\mathcal{D}} \{px\},$$

for all $x \in \mathcal{C}$.

Proof. We will induct on K. The base case is where $K = \{c\}$. In this case the left hand side is $\mathrm{Maps}_{\mathcal{C}}(c, x)$ and the right hand side is $\mathrm{Maps}(pc, px)$. Hence we obtain an equivalence by assumption that $\mathcal{C} \simeq \mathcal{D}$. For the inductive step consider the pushout

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, "\bar{f}"] \arrow[d] & K_\alpha \arrow[d, "f"] \\
\Delta[n] \arrow[r, "f"] & K_{\alpha+1}
\end{tikzcd}$$

Write $C_\alpha = \mathcal{C}^{j_\alpha/} \times_{\mathcal{C}} \{x\}$ and $D_\alpha = \mathcal{D}^{j_\alpha \circ p/} \times_{\mathcal{D}} \{px\}$, where j_α is the restriction of j to $K_\alpha \to K$. Now

$$\mathcal{C}^{j_{\alpha+1}/} = \mathcal{C}^{j_\alpha/} \times_{\mathcal{C}}^{\partial\Delta[n]/} \mathcal{C}^{\Delta[n]/}.$$

This implies that $C_{\alpha+1}$ is the pullback

$$\begin{tikzcd}
C_{\alpha+1} \arrow[r] \arrow[d] & C_\alpha \arrow[d] \\
C_f \arrow[r] & C_{f|_{\partial\Delta[n]}}
\end{tikzcd}$$

Here $C_f = \mathcal{C}^{\Delta[n]/} \times_{\mathcal{C}} \{x\}$. Now since $\mathcal{C}^{\Delta[n]/} \to \mathcal{C}^{\partial\Delta[n]/}$ is a left fibration we find that the bottom arrow in the square above is a left fibration. However, both the source and the target are Kan complexes, whence the arrow is in fact a Kan fibration (this is a lemma we will not prove—it is just a parameterized version of something we have already proven). Draw the same diagram for D

$$\begin{tikzcd}
D_{\alpha+1} \arrow[r] \arrow[d] & D_\alpha \arrow[d] \\
D_f \arrow[r] & D_{f|_{\partial\Delta[n]}}
\end{tikzcd}$$

and note that by the inductive step $C_\alpha \simeq D_\alpha$ and $C_f \simeq D_f$. By induction we get equivalences which induce an equivalence $C_{\alpha+1} \simeq D_{\alpha+1}$ because the bottom arrows are Kan fibrations. □

We will implicitly regard 1-categories as ∞-categories via the nerve.

14.2. Back to factorization homology.

Theorem 60. *Let Spaces be the ∞-category of spaces and $\mathscr{D}\mathsf{isk}_{n/M}$ be the (nerve of) category of n-disks. Then*

$$\int_M \mathrm{id} := \operatorname{colim}\left(\mathscr{D}\mathsf{isk}_{n/M} \xrightarrow{\mathrm{id}} \mathsf{Spaces}\right) \simeq M$$

Definition 61. An ∞-category is **κ-filtered**, for κ some ordinal, if for any κ-small $\mathcal{K}$ together with a functor $\mathcal{K} \to \mathcal{C}$, there exists a factorization

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{K}^{\triangleright} \arrow[ur, dashed] & \end{tikzcd}$$

For some intuition recall the analogous 1-categorical definition.

Definition 62. For $\mathcal{C}$ an ordinary category, we say that C is **filtered** if

(1) for any finite set $\{x_i\}$ of objects there exists an x such that there is a map $x_i \to x$ for all i,
(2) and for all $f, g : x \rightrightarrows y$ there exists $h : y \to z$ such that $hf = hg$.

Example 63. As a simple example consider the category of natural numbers $\mathbb{N}$ with unique morphisms $m \to n$ when $m \leq n$. Condition (1) is clear and condition (2) is trivial due to the hom-sets being (at most) one-element sets.

Example 64. If $\mathcal{C}$ has a final object then $\mathcal{C}$ is filtered.

Example 65. Consider the subposet of open subsets of M that contain a fixed point $p \in M$, $\mathsf{Opens}(M)_p^{\mathrm{op}} \subset \mathsf{Opens}(M)^{\mathrm{op}}$. Notice that $\{p\}$ wants to be a final object, but it need not be open. Let's check that this is filtered. For any finite collection $\{U_i\}$ the intersection $x = \cap_I U_i$, gives us the first condition. Now suppose we have $f, g : U_i \to U_j$. In this case we just let $z = U_j$: our category is a poset so $f = g$ already. Hooray.

Lemma 66. *If $\mathcal{C}$ is filtered as a 1-category then it is filtered as an ∞-category.*

To see this, notice that any category can be built as a colimit in Cat from one-point categories.

Proof idea. Induct on $\mathcal{K}$, building by coproducts and equalizers. $\square$

Filtered categories are nice because colimits indexed by them enjoy good properties.

Lemma 67 (HTT 5.3.1.20)**.** *If $\mathcal{C}$ is filtered then $B\mathcal{C} \simeq *$.*

Recall that we are speaking model-independently—B is the left adjoint to the inclusion $\mathsf{Spaces} \hookrightarrow \mathsf{Cat}_\infty$. In particular, given a map $\mathcal{C} \to X$ for X a space it factors uniquely up to homotopy through the classifying space.

If we think of quasicategories, it is the left-adjoint of the inclusion $\mathsf{Kan} \hookrightarrow \mathsf{QCat}$. What does it do? Well it is a colimit-preserving functor, so we should describe it on the building blocks $\Delta[n]$. By definition $\Delta[n] = N_\bullet([n])$ where by $[n]$ we think of the poset as a category. Define $[n]'$ to be the smallest groupoid with objects that of $[n]$. As a category, $[n]' \simeq *$. Recall that $\Delta[n]$ is a quasicategory but not a Kan complex. We send $\Delta[n]$ to $N_\bullet([n]')$.

Wouldn't this just be the discrete groupoid?

15. Computing factorization homology [10/25/17]

Goette, Igusa, Williams have a theorem (in the stable range): you get all exotic bundle structures through Hatcher's construction. This is related to factorization homology. Anyway, back to filtered ∞-categories.

Proof of HTT 5.3.1.20. This proof will be model-dependent. Let $\mathcal{C}$ be a quasi-category. Then $B\mathcal{C} = |\mathcal{C}|$, the geometric realization. Take any finite subcomplex $K \hookrightarrow |\mathcal{C}|$. This is represented by a map $\bar{K} \to \mathcal{C}$ where $\bar{K}$ is a simplicial set such that $|\bar{K}| = K$. Since $\mathcal{C}$ is filtered, there exists a factorization of this map through $\bar{K}^{\triangleright} \to \mathcal{C}$. Taking geometric realizations we have

$$\begin{tikzcd} K \arrow[r] \arrow[d] & {|\mathcal{C}|} \\ {* \simeq |\bar{K}^{\triangleright}|} \arrow[ur, dashed] & \end{tikzcd}$$

which implies that any finite subcomplex of $|\mathcal{C}|$ is contractible. We conclude that $|\mathcal{C}|$ is contractible. $\square$

Theorem 68. *Consider the identity functor* $\mathrm{id} : \mathsf{Disk}_n \to \mathsf{Spaces}$. *Then*

$$\int_M \mathrm{id} \simeq M.$$

Proof. We use a hypercover argument. In particular, we will use without proof (for now) that given a functor $\mathcal{C} \subset \mathsf{Opens}(M) \xrightarrow{\mathrm{id}} \mathsf{Spaces}$ such that $B\mathcal{C}_x \simeq *$ for all $x \in M$ then

$$\mathrm{hocolim}_{\mathcal{C}}\, \mathrm{id} \simeq M.$$

Apply this fact to $\mathcal{C} = \mathsf{Disk}_{n/M} \to \mathsf{Opens}(M)$. We have to show that $B(\mathsf{Disk}_{n/M})_x \simeq *$. We will show that $(\mathsf{Disk}_{n/M})_x$ is cofiltered. In particular, we need to show that for any finite collection $\{U_i \ni x\}$ there exists U with $U \to U_i$. Take any $U \cong \mathbb{R}^n \subset \cap_i U_i$ containing x. Then we need to show that for any two maps $U_i \to U_j$, there is a map $U \to U_i$ equalizing them. But this is clear because we are in a poset so there is only one map from U_i to U_j Now, using the fact that $B\mathcal{C} \simeq B\mathcal{C}^{\mathrm{op}}$ (alternatively repeat above arguments with cofiltered and left cones), we are done by the above lemma. $\square$

This is not true unless we are looking at the image of the n-disks.

Definition 69. We say that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is an adjunction if there are natural equivalences

$$\mathrm{Maps}_{\mathcal{D}}(Fx, y) \simeq \mathrm{Maps}_{\mathcal{C}}(x, Gy).$$

Proposition 70. *Left adjoints preserve colimits and right adjoints preserve limits.*

Proof. The proof is the same as the 1-categorical case. Suppose we have $j : J \to \mathcal{C}$. We want to verify that if $\mathrm{colim}_j \simeq x$ then $\mathrm{colim}_J F \circ j \simeq F(x)$. In other words, given $\mathcal{C}^{j/} \simeq \mathcal{C}^{x/}$ we want to show that $\mathcal{D}^{F \circ j/} \simeq \mathcal{D}^{F(x)/}$. But the maps from these latter two to $\mathcal{D}$ is a left fibration whence it suffices to show that the fibers are equivalent: for all $d \in \mathcal{D}$,

$$\mathrm{Maps}_{\mathcal{D}}(F(x), d) \simeq \mathrm{Maps}_{\mathrm{Fun}(J,\mathcal{D})}(F \circ j, \underline{d}).$$

By our adjunction,

$$\mathrm{Maps}_{\mathcal{D}}(F(x), d) \simeq \mathrm{Maps}_{\mathcal{C}}(x, G(d)) \simeq \mathrm{Maps}_{\mathrm{Fun}(J,\mathcal{C})}(j, \underline{G(d)}).$$

But

$$\mathrm{Maps}_{\mathrm{Fun}(J,\mathcal{C})}(j, \underline{G(d)}) \simeq \lim_{z\in J^{\mathrm{op}}} \mathrm{Maps}_{\mathcal{C}}(jz, Gd)$$

and

$$\mathrm{Maps}_{\mathrm{Fun}(J,\mathcal{D})}(F\circ j, \underline{d}) \simeq \lim_{z\in J^{\mathrm{op}}} \mathrm{Maps}_{\mathcal{D}}(Fjz, d),$$

which are the same by our adjunction.

Notice that this is a model independent proof. □

The following is an important adjunction (of ∞-categories) to keep in mind. The singular chains $C_* : \mathsf{Spaces} \to \mathsf{Ch}$ has a right adjoint G such that $\pi_* GV$ is $H_* V$ for $* \geq 0$ and 0 otherwise. This comes from the adjunction between sSet and sAb given by free abelian group and forgetful funtors. Hence C_* preserves (homotopy) colimits.

Notice that there exists a unique colimit preserving functor $F : \mathsf{Spaces} \to \mathcal{V}$ for $\mathcal{V}$ any ∞-category with colimits with $F(*) = v$ since any space is built as a colimit of contractible spaces. In particular, any $X = \mathrm{colim}(\mathcal{U} \xrightarrow{j} \mathsf{Spaces})$ where $u \mapsto ju \simeq *$, which yields the same colimit as the constant diagram $\mathcal{U} \to \mathcal{V}$ and $F(X) = \mathrm{colim}(\mathcal{U} \to \mathcal{V})$.

Let's do another calculation. Consider the functor $\mathbb{Z}^{\oplus} : \mathsf{Disk}_n \to \mathsf{Ch}$ sending $\coprod_I \mathbb{R}^n \mapsto \mathbb{Z}^{\oplus I}$. Let's calculate $\int_M \mathbb{Z}^{\oplus}$. Notice first that $\mathbb{Z}^{\oplus I} \simeq C_*(\coprod_I \mathbb{R}^n)$. Hence

$$\int_M \mathbb{Z}^{\oplus} \simeq \int_M C_* \circ \mathrm{id} = \mathrm{colim}\left(\mathsf{Disk}_{n/M} \xrightarrow{\mathrm{id}} \mathsf{Spaces} \xrightarrow{C_*} \mathsf{Ch}\right).$$

Since C_* is colimit preserving,

$$\int_M \mathbb{Z}^{\oplus} \simeq C_* \,\mathrm{colim}\left(\mathsf{Disk}_{n/M} \xrightarrow{\mathrm{id}} \mathsf{Spaces}\right) \simeq C_* \int_M \mathrm{id} \simeq C_* M.$$

Similarly for those of you familiar with spectra, there is a adjunction of spaces with spectra given by Ω^∞ and Σ^∞_* and exactly the same proof shows that

$$\int_M \mathbb{S}^{\oplus} \simeq \Sigma^\infty_* M.$$

Definition 71. Let $\mathsf{Disk}_n^{=1} \subset \mathsf{Disk}_n$ be those n-disks with $|I| = 1$.

Now repeating the arguments of this lecture for this subcategory, we find

$$\mathrm{colim}\left({\mathsf{Disk}_n^{=1}}_{/M} \xrightarrow{\mathrm{id}} \mathsf{Spaces}\right) \simeq M.$$

In other words, it is enough to probe manifolds with single-opens. However, all of this seeming differential topology washes out.

Proposition 72. *There is an equivalence of ∞-categories ${\mathscr{D}\mathsf{isk}_n^{=1}}_{/M} \simeq M$. Or as quasicategories, ${\mathscr{D}\mathsf{isk}_n^{=1}}_{/M} \simeq \mathrm{Sing}(M)$.*

Proof sketch. We construct a functor ev_0 given by evaluation at 0 sending $\phi : \mathbb{R}^n \hookrightarrow M \mapsto \phi(0) \in M$. This functor induces an equivalence on objects so it remains to check on mapping spaces. In particular, need to check $\mathrm{Maps}_{\mathscr{D}\mathsf{isk}_{n/M}}(\mathbb{R}^n, \mathbb{R}^n) \simeq \Omega M$. □

Now using that $\mathsf{Disk}_{n/M} \to \mathscr{D}\mathsf{isk}_{n/M}$ is a localization, we find that $\mathrm{colim}_M * \simeq M$.

16. Nonabelian Poincaré duality I [10/27/2017]

Example 73. Recall we have the free E_n-algebra on a space Z, $A = F(Z)$. This functor $A : \mathsf{Disk}_n \to \mathsf{Spaces}$ sends

$$U \mapsto \coprod_{i \geqslant 0} \mathrm{Conf}_i(U) \times_{\Sigma_i} Z^i.$$

Proposition 74. *The factorization homology of A is*

$$\int_M F(Z) = \coprod_{i \geqslant 0} \mathrm{Conf}_i(M) \times_{\Sigma_i} Z^i.$$

We will need the following lemma.

Lemma 75. *The map from the homotopy colimit*

$$\operatorname*{colim}_{U \in \mathsf{Disk}_{n/M}} \mathrm{Conf}_i(U) \xrightarrow{\sim} \mathrm{Conf}_i(M)$$

is a homotopy equivalence.

Proof. We have $\mathsf{Disk}_{n/M} \hookrightarrow \mathsf{Opens}(M) \xrightarrow{\mathrm{Conf}_i} \mathsf{Spaces}$ which we can look at as

$$\mathsf{Disk}_{n/M} \xrightarrow{\mathrm{Conf}_i} \mathsf{Opens}(\mathrm{Conf}_i(M)) \xrightarrow{\mathrm{id}} \mathsf{Spaces}.$$

Apply our hypercover lemma to this line: we need to check that for each $\{x_1, \ldots, x_i\} \in \mathrm{Conf}_i(M)$, $\mathsf{Disk}_{n/M,\{x_1,\ldots,x_i\}}$ has a contractible classifying space. We have already done this in the case when $i = 1$. In particular, we just check that the category is cofiltered—perform the argument before for each point. Let's check the two conditions: given any finite collection of U_j containing the $x_1, \ldots, x_i$, there exists $U \subset \cap_j U_j$ (the intersection itself is not a disjoint union of disks) with $U \to U_j$ for all j. The second condition is again trivial by the fact that our category is a poset. □

Notice that this lemma holds for $\mathrm{Conf}_i(U) \times_{\Sigma_i} Z^i$, which we will use below. Alternatively one might use the fact that products commute with colimits. In particular $\operatorname{colim}_J Z \times F_j \to Z \times \operatorname{colim}_{j \in J} F_j$ is an equivalence.

Proof of proposition. Formally, colimits commute so we have

$$\begin{aligned}
\operatorname*{colim}_{U \in \mathsf{Disk}_{n/M}} \coprod_{i \geqslant 0} \mathrm{Conf}_i(M) \times_{\Sigma_i} Z^i &= \coprod_{i \geq 0} \operatorname*{colim}_{U \in \mathsf{Disk}_{n/M}} \left(\mathrm{Conf}_i(U) \times_{\Sigma_i} Z^i\right) \\
&\simeq \coprod_{i \geqslant 0} \left(\operatorname*{colim}_{U \in \mathsf{Disk}_{n/M}} \mathrm{Conf}_i(U) \right) \times_{\Sigma_i} Z^i \\
&\simeq \coprod_{i \geqslant 0} \mathrm{Conf}_i(M) \times_{\Sigma_i} Z^i,
\end{aligned}$$

as desired. □

This proof is written down in the paper [AF15] of Ayala and F.

Let's turn to a more complicated example. Recall that we have a functor $\mathsf{Disk}_n \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Spaces}$ for Z a pointed space. We have a natural map

$$\operatorname{colim}\left(\mathsf{Disk}_{n/M} \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Spaces}\right) \to \operatorname{colim}\left(\mathcal{M}\mathsf{fld}_{n/M} \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Spaces}\right)$$

identifying the n-disks as manifolds. Notice that the colimit on the right hand side has a category with a final object as its source, whence the colimit is equivalent to $\mathrm{Maps}_c(M, Z)$. This leads us to the following theorem, due to Segal, Salvatore, and Lurie, in various formulations.

Theorem 76 (Nonabelian Poincaré duality)**.** *Let M be an n-manifold and Z be an $(n-1)$-connected pointed space. Then the map defined above*

$$\int_M \mathrm{Maps}_c(-, Z) \xrightarrow{\sim} \mathrm{Maps}_c(M, Z)$$

is an equivalence.

The name comes from the fact that if one takes Z to be an Eilenberg-Maclane space, then recover the usual statement of Poincaré duality.

Remark 77. Let's see why we need the condition that $\pi_* Z = 0$ for $* < n$. Let's first consider the case where $Z = S^0$. Let's look at the left-hand side:

$$\operatorname*{colim}_{U \in \mathsf{Disk}_{n/M}} \mathrm{Maps}_c(U, S^0) \simeq B(\mathsf{Disk}_{n/M})$$

which is connected because between any two objects there is a morphism. Now let's look at $\mathrm{Maps}_c(M, S^0)$ for M compact. But now $\mathrm{Maps}_c(M, S^0) = \mathrm{Maps}(M, S^0)$ has at least 2 components for $M \neq \varnothing$.

More formally, note that $\mathrm{Maps}_c(\mathbb{R}^n, Z) \simeq \Omega^n Z$. But using the long exact sequence on the homotopy groups of a fibration repeatedly, we compute

$$\pi_* \Omega^n Z = \pi_{*+n} Z \quad \text{for} \quad * \geqslant 0.$$

In particular, this does not depend on $\pi_* Z$ for $* < n$. So this statement could not possibly be true because we could just change the space such that only the $< n$ homotopy groups change. In particular, take $\tau^{\geq n} Z \xrightarrow{\text{hofiber}} Z \to P_{n-1} Z$ where $P_{n-1} Z$ is the $n-1$ Postnikov stage:

$$\mathrm{Maps}_c(\mathbb{R}^n, Z) \simeq \mathrm{Maps}_c(\mathbb{R}^n, \tau^{\geq n} Z).$$

So the functor on the left will not change, but the functor on the right can always detect these by choosing M appropriately. This tells us that we had better choose $Z \simeq \tau^{\geq n} Z$.

Grisha: does the left hand side depend on the map $M \to BO(n)$. John: No, in particular $\mathsf{Disk}_{n/M}$ only knows about the homeomorphism type of M. Morevoer there is a cancellation of sorts $\mathsf{Disk}^{\mathrm{fr}}_{n/M} \simeq \mathsf{Disk}_{n/M}$. Again, this is something you can find in the paper [AF15].

We can think of both sides as functors out of $\mathcal{M}\mathsf{fld}_n$. Given a natural transformation between to functors out of here, to show it is an equivalence, we might first show that it is an equivalence on $\mathbb{R}^n$, and then show that the functors satisfy the same "gluing/cosheaf" properties. In particular, do these functors satisfy something like Mayer-Vietoris?

To prove such a property, one might "filter" either Z or M. Jacob, in his book, does the former (via Postnikov stages), while John does it by breaking down M. We'll be doing the latter.

Definition 78. For $\mathcal{V}$ a symmetric monoidal ∞-category, a **homology theory for n-manifolds valued in $\mathcal{V}$** is a symmetric monoidal functor

$$F : (\mathcal{M}\mathsf{fld}_n, \coprod) \longrightarrow (\mathcal{V}, \otimes)$$

such that whenever $M \cong M' \coprod_{M_0 \times \mathbb{R}} M''$ for $\dim M_0 = n-1$ then

$$F(M') \otimes_{F(M_0 \times \mathbb{R})} F(M'') \to F(M)$$

is an equivalence.

Theorem 79. *If we write the collection $\mathcal{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{V})$ for all such homology theories, then there is an equivalence*

$$\begin{tikzcd} \mathcal{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{V}) \arrow[r, bend left, "\mathrm{ev}_{\mathsf{Disk}_n}"] & \mathsf{Alg}_{\mathsf{Disk}_n}(\mathcal{V}) \arrow[l, bend left, "\int_{-}"] \end{tikzcd}$$

Theorem 80. $\mathrm{Maps}_c(-, Z)$ *and* $\int_{-} \mathrm{Maps}_c(-, Z)$ *are homology theories valued in* Spaces.

17. Nonabelian Poincaré duality II [10/30/2017]

Let $\mathcal{V}$ be a symmetric monoidal ∞-category such that $v\otimes - : \mathcal{V}\to\mathcal{V}$ distributes over all (sifted) colimits, i.e.

$$\operatorname*{colim}_J (v\otimes F_j) \xrightarrow{\cong} v\otimes \operatorname*{colim}_J F_j$$

for all $F : J\to\mathcal{V}$. In this context we stated the equivalence above. First of all: how do we even know that the thing on the left hand side of our excision statement, $F(M')\otimes_{F(M_0\times\mathbb{R})} F(M'')$, even makes sense?

Definition 81. Define $\mathscr{M}\mathsf{fld}_n^\partial$ to be the topological category with objects manifolds possibly with boundary, and morphisms open embeddings. In particular, boundaries must be sent to boundaries.

We will construct $\Delta^{\mathrm{op}}\to\mathscr{M}\mathsf{fld}_{n/M}$ given a gluing $M\cong M'\cup_{M_0\times\mathbb{R}} M''$. Then, given a functor $F:\mathscr{M}\mathsf{fld}_n\to\mathcal{V}$ (or into $(\mathsf{Spaces},\times)$), we can define

$$F(M')\otimes_{F(M_0\times\mathbb{R})} F(M'') := \operatorname*{colim}_{\Delta^{\mathrm{op}}} F$$

Fix this diagram

as a colimit over the composite. Suppose and $M|_{(-1,1)}\cong M_0\times(-1,1)$. Here we are using the **homework** fact that $N\cong N\cup_{\partial N}\partial N\times[0,1)$.

Fix the notation here, make the D cal

Consider now the functor of topological categories (a bit subtle actually)

$$\mathsf{Disk}^\partial_{1/[-1,1]} \xrightarrow{\pi^{-1}} \mathscr{M}\mathsf{fld}_{n/M}$$

sending $U\hookrightarrow[-1,1]\mapsto \pi^{-1}U$.

Definition 82. We define Disk_n^∂ to be the subcategory of $\mathscr{M}\mathsf{fld}_n^\partial$ consisting of $\coprod_U \mathbb{R}^n \coprod_j \mathbb{R}^n_{\geq 0}$. Notice it doesn't contain the disk itself.

Lemma 83 (AF 3.11)**.** *There exists a functor $\Delta^{op}\to\mathsf{Disk}^{\partial,or}_{1/[-1,1]}$ which is final.*

Proof. Define $\mathcal{S}\subset\mathsf{Disk}^{\partial,\mathrm{or}}_{1/[-1,1]}$ to be the subcategory of objects $U\subset[-1,1]$ such that $\{-1,1\}\subset U$ and

$$U=[0,\varepsilon)\coprod\mathbb{R}^{\amalg l}\coprod(\delta,1]\hookrightarrow[-1,1].$$

This gives us a functor $\mathcal{S}\to\Delta^{\mathrm{op}}$ sending $U\mapsto\pi_0([-1,1]\setminus U)$, i.e. counting the gaps (order it left to right). In fact we claim that $S\simeq\Delta^{\mathrm{op}}$. This is an obvious bijection on the collections of objects. We need to check that the spaces of maps

$$\mathsf{Disk}^{\partial,\mathrm{or}}_{1/[-1,1]}([0,\varepsilon)\coprod\mathbb{R}^{\amalg i}\coprod(\delta,1],[0,\varepsilon')\coprod\mathbb{R}^{\amalg i'}\coprod(\delta',1])\simeq\Delta([i'],[i]).$$

Concretely, let's check that

$$\mathcal{M}\mathsf{fld}^{\partial,\mathrm{or}}_{1/[-1,1]}(\mathbb{R},\mathbb{R})\simeq *.$$

But this is the homotopy pullback

$$\begin{tikzcd}
\mathcal{M}\mathsf{fld}^{\partial,\mathrm{or}}_{1/[-1,1]}(\mathbb{R},\mathbb{R}) \arrow[r] \arrow[d] & \mathcal{M}\mathsf{fld}^{\partial,\mathrm{or}}_{1}(\mathbb{R},\mathbb{R}) \arrow[d] \\
* \arrow[r] & \mathrm{Maps}(\mathbb{R},[-1,1])
\end{tikzcd}$$

by definition of ∞-categorical overcategories. But the two spaces on the right hand side are contractible, hence we obtain what we wanted. This implies the discreteness and hence the equivalence of the mapping spaces.

Let us check that the functor is final. We use Quillen's theorem A. In particular, we show that $B(\mathcal{S}^{V/} \simeq *$ for $V \in \mathsf{Disk}^{\partial,\mathrm{or}}_{1/[-1,1]}$. We have

$$V = [0,\varepsilon)^{\amalg ?} \coprod \mathbb{R}^{\amalg i} \coprod (\delta, 1)^{\amalg ?}.$$

If $V \in \mathcal{S}$ then $\mathcal{S}^{V/}$ has an initial object whence $B\mathcal{S}^{V/} \simeq *$. On the other hand, if $V \in \mathcal{S}$ has neither 0 nor 1 then

$$V' = V \coprod [0,\varepsilon) \coprod (\delta, 1]$$

is initial in $\mathcal{S}^{V/}$. Indeed, we claim that given any $W \subset [-1,1]$ and an embedding $V \hookrightarrow W$, the space of factorizations through V' is contractible. But this is just that

$$\mathcal{M}\mathsf{fld}_{/[-1,1]}([0,\varepsilon), [0,\varepsilon')) \simeq *.$$

Hence $\mathcal{S}$ is final. $\square$

Now, given $M' \cup_{M_0 \times \mathbb{R}} M''$, and a functor $F : \mathscr{M}\mathsf{fld}_n \to \mathcal{V}$, we define

$$F(M') \otimes_{F(M_0 \times \mathbb{R})} F(M'') = \operatorname*{colim}_{\Delta^{\mathrm{op}}} F$$

where the functor is the composite

$$\Delta^{\mathrm{op}} \to \mathcal{M}\mathsf{fld}^{\partial,\mathrm{or}}_{1/[-1,1]} \to \mathscr{M}\mathsf{fld}_n \to \mathcal{V},$$

which if F is symmetric monoidal then this is just a two-sided bar construction.

Sam: why did we do this construction instead of just writing it as a bar construction in the first place? John: this is not just a normal simplicial object. It's a simplicial object in an ∞-category. Indeed, there is no functor $\Delta^{\mathrm{op}} \to \mathsf{Mfld}_{n/M}$. We are useful a flexibility of isotopies and contractibilities to actually build this object.

Now, if A is an n-disk algebra in $\mathcal{V}$, we would like to show that $\int_{\_} A$ is a $\otimes$-homology theory, i.e.

$$\int_{M'} A \otimes_{\int_{M_0 \times \mathbb{R}} A} \int_{M''} A \simeq \int_M A.$$

In fact we will prove something stronger. We will write down a pushforward formula. Say we are given $f : M \to (N, \partial N)$ with $M|_{int(N)} \to int(N)$ and $M|_{\partial N} \to \partial N$ fiber bundles.

Theorem 84. *Given the pushforward*

$$f_* A : \mathsf{Disk}^{\partial}_{n/N} \xrightarrow{f^{-1}} \mathscr{M}\mathsf{fld}_{n/M} \xrightarrow{\int A} \mathcal{V},$$

there is an equivalence

$$\int_M A \simeq \int_N f_* A.$$

This will imply that factorization homology is a $\otimes$-homology theory.

18. Nonabelian Poincaré duality III [11/01/2017]

We were proving a pushforward formula for factorization homology. Recall that we have a map $f : M \to N \supset \partial N$ with the restriction of f to the interior and boundary of N both being fiber bundles.

fix this diagram

Definition 85. We define Disk_f to be the limit of the diagram i.e. triples consisting of k-disks $U \subset N$, n-disks $V \subset M$, with $V \hookrightarrow f^{-1}U$.

Lemma 86. *The map* $\mathsf{Disk}_f \xrightarrow{\mathrm{ev}_0} \mathsf{Disk}_{n/M}$ *sending* $(U, V, V \hookrightarrow f^{-1}U) \mapsto V$ *is final.*

This lemma is important because it means that the factorization homology can be computed on Disk_f. Also, looking back at this, it's way more technical than John remembers... he thought that that was his least technical paper... and maybe it is—just everything he writes is incomprehensible.

Lemma 87. *The map* $\mathsf{Disk}_{n/M} \to \mathcal{D}\mathsf{isk}_{n/M}$ *is a localization.*

Corollary 88. *The map above is final, whence* $\int_M A$ *is a colimit over either* $\mathsf{Disk}_{n/M}$ *or* $\mathcal{D}\mathsf{isk}_{n/M}$*.*

Corollary 89. *For any functor* $A : \mathsf{Disk}_n \to \mathcal{V}$*,*

$$\int_M A \simeq \operatorname{colim}\left(\mathsf{Disk}_f \xrightarrow{\mathrm{ev}_0} \mathcal{D}\mathsf{isk}_{n/M} \xrightarrow{A} \mathcal{V}\right).$$

Theorem 90. *We have the pushforward formula*

$$\int_M A \simeq \int_N f_* A$$

where $f_*A : \mathcal{D}\mathsf{isk}^{\partial}_{k/N} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}_{n/M} \xrightarrow{\int A} \mathcal{V}$*.*

We have

$$\begin{tikzcd}
\mathsf{Disk}_f \arrow[r, "\mathrm{ev}_0"] \arrow[d] & \mathcal{D}\mathsf{isk}_{n/M} \arrow[r, "A"] & \mathcal{V} \\
\mathsf{Disk}^{\partial}_{k/N} \arrow[d, "q"] & & \\
* & &
\end{tikzcd}$$

We note that

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_{n/M} \xrightarrow{A} \mathcal{V}\right) \simeq \operatorname{colim}(\mathcal{D}\mathsf{isk}_f \xrightarrow{A \circ \mathrm{ev}_0} \mathcal{V}).$$

If we call the vertical composite p, the left-hand side is just $\mathsf{LKan}_p(A \circ \mathrm{ev}_1$ since colimits are just left Kan extensions to the terminal ∞-category. But

$$\mathsf{LKan}_p(A \circ \mathrm{ev}_0) \simeq \mathsf{LKan}_q \mathsf{LKan}_{\mathrm{ev}_1}(A \circ \mathrm{ev}_0).$$

Now we need to figure out how to evalute the left Kan extension at 1. Let's take a brief vacation from the proof.

The evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1$ from the category $\mathrm{Ar}(\mathcal{D})$ to $\mathcal{D}$ have some nice properties. The fiber over some object $d \in \mathcal{D}$ is just the undercategory $\mathcal{D}^{d/}$. Consider

$$\mathcal{D}^{d/} \hookrightarrow \mathrm{Ar}(\mathcal{D})^{\mathrm{ev}_0} \times_{\mathcal{D}} \mathcal{D}^{d/} = \mathrm{Ar}(\mathcal{D})^{d/}.$$

This map is a left adjoint. The right adjoint is simple: just compose. On the other hand, John: check this for yourself.

$$\mathcal{D}_{/d} \hookrightarrow \mathrm{Ar}(\mathcal{D})^{\mathrm{ev}_1} \times_{\mathcal{D}} \mathcal{D}_{/d}$$

is a right adjoint (again the other is given by composition). Now recall that left adjoints are initial functors and right adjoints are final. Observe that given $\pi : \mathcal{A} \to \mathcal{D}$ such that $\mathcal{A}_d \hookrightarrow \mathcal{A}_{/d} = \mathcal{A} \times_{\mathcal{D}} \mathcal{D}_{/d}$ is final for any d, then

$$\mathsf{LKan}_\pi F \simeq \operatorname*{colim}_{\mathcal{A}_{/d}} F \simeq \operatorname*{colim}_{\mathcal{A}_d} F.$$

Definition 91. Such a functor π is called a **locally Cartesian fibration**.

What we have seen above is that evaluation at 0 and 1 are locally coCartesian and Cartesian fibrations.

Returning from our vacation (which are never as relaxing as you expect) we find that by the Cartesian fibration property,

$$\mathsf{LKan}_{\mathrm{ev}_0}(A \circ \mathrm{ev}_1)(U) \simeq \mathrm{colim}(\mathsf{Disk}_f|_U \xrightarrow{A \circ \mathrm{ev}_0} \mathcal{V}).$$

But $\mathsf{Disk}_f|_U = \mathsf{Disk}_{n/f^{-1}U}$. Hence we find

$$\mathsf{LKan}_{\mathrm{ev}_0}(A \circ \mathrm{ev}_1)(U) = \mathrm{colim}(\mathsf{Disk}_{n/f^{-1}U} \xrightarrow{A} \mathcal{V}) = \int_{f^{-1}U} A =: f_*A(U).$$

But now $\int_M A \simeq \int_N f_*A$, which completes the proof (using the finality of the localization).

There is a bit of a gap here: we used the locally Cartesian fibration property. However, really what we had was a pullback of a locally Cartesian fibration property. Unfortunately, this property is not preserved under pullbacks. However we are ok because we have Cartesian, not just locally Cartesian.

What's left is two lemmas. The localization result and the finality result.

Lemma 92. *The map* $\mathrm{ev}_0 : \mathcal{D}\mathsf{isk}_f \to \mathcal{D}\mathsf{isk}_{n/M}$ *is final.*

Proof. We use Quillen's theorem A, which says that $\mathcal{F} \to \mathcal{D}$ is final if $B(\mathcal{F}^{d/}) \simeq *$ for all $d \in \mathcal{D}$. So for any $V \in \mathcal{D}\mathsf{isk}_{n/M}$ we need to check that $B(\mathcal{D}\mathsf{isk}_f^{V/}) \simeq *$. Recall that ev_0 is a (locally) Cartesian fibration. This means that the fiber of ev_0 included into the undercategory

$$\mathcal{D}\mathsf{isk}_f|_V \hookrightarrow \mathcal{D}\mathsf{isk}_f^{V/}$$

is a left adjoint. Now we recall that if we are given an adjunction between $\mathcal{C}$ and $\mathcal{D}$ by F and G. This implies an equivalence of classifying spaces $B\mathcal{C} \simeq B\mathcal{D}$ (just a special case of being final or initial since the classifying space functor is a colimit of the constant functor; alternatively just look at how the unit/counit maps give you homotopies). This implies that we can show instead that the fiber over V, $\mathcal{D}\mathsf{isk}_f|_V$, has contractible classifying space. This category consists of k-disks $U \subset N$ with $V \hookrightarrow f^{-1}U$. We don't have much time left, but the basic idea is to use the hypercover argument. □

Yajit: how to make the "colliding points" picture of factorization homology precise? John: consider the subcategory $\mathcal{D}\mathsf{isk}_{n/M}^{\mathrm{surj}} \subset \mathsf{Disk}_{n/M}$ which is surjective on π_0. This subcategory is the exit-path infinity category of the Ran space $\mathsf{Ran}(M)$. But $\mathrm{Fun}(\mathrm{Exit}(\mathsf{Ran} M), \mathcal{V})$ are constructible cosheaves on $\mathsf{Ran}(M)$, and then $\int_M A = \Gamma(\mathsf{Ran}, A)$.

19. Nonabelian Poincaré duality IV [11/03/2017]

Proposition 93 (AF 2.19)**.** *The map $\mathsf{Disk}_{n/M} \to \mathcal{D}\mathsf{isk}_{n/M}$ is a localization, i.e. $\mathcal{D}\mathsf{isk}_{n/M}$ is the universal ∞-category under $\mathsf{Disk}_{n/M}$ such that isotopy equivalences to equivalence. More precisely, for any functor $\mathsf{Disk}_{n/M} \to \mathcal{C}$ sending isotopy equivalences to equivalences, there exists a unique factorization through $\mathcal{D}\mathsf{isk}_{n/M}$.*

There are two parts: a formal criterion for localizations, and a topological argument via hypercovers. Let's look at the former. Here's a naive guess: a functor $\mathcal{C} \to \mathcal{D}$ a localization with respect to $\mathcal{I} \subset \mathcal{C}$ if the space of objects of $\mathcal{D} = B\mathcal{I}$. If, say $\mathcal{I}$ was the underlying groupoid of $\mathcal{C}$ (the trivial localization), then this checks out. Moreover, for the space of morphisms take $B\left(\mathrm{Fun}^{\mathcal{I}}([1], \mathcal{C}\right)$. Likewise for p-tuples (not just spans). This is a sufficient condition.

Proof. We check that the space of objects is that of a localization. The rest can be found in the paper. In particular we check that

$$B\,\mathrm{Fun}^{\text{iso eq}}([p], \mathsf{Disk}_{n/M}) \simeq \mathrm{Maps}([p], \mathcal{D}\mathsf{isk}_{n/M})$$

for $p = 0$. In this case the left hand side is $\mathsf{Disk}_{n/M}^{\text{iso eq}}$ which is just $\coprod_{k\geq 0}(\mathsf{Disk}_{n/M}^{=k})^{\text{iso eq}}$ and the right hand side is the underlying groupoid $\mathcal{D}\mathsf{isk}_{n/M}$ which is just $\coprod_{k\geq 0}(\mathcal{D}\mathsf{isk}_{n/M}^{=k})^0$.

Change the underlying groupoid notation.

Let's start with the right hand side. For $k = 1$ it has objects $\mathbb{R}^n \hookrightarrow M$ so $\mathcal{D}\mathsf{isk}_{n/M}^{=1} \simeq M$ where we think of M as an ∞-groupoid. The functor giving the equivalence is given by evaluation at 0 (one needs to actually check that this is a functor). To check that it's an equivalence we check that

$$\begin{aligned}\mathrm{Emb}_{/M}(\phi : \mathbb{R}^n \to M, \psi : \mathbb{R}^n \to M) &\simeq \mathrm{Maps}_M(\phi(0), \psi(0)) \\ &= \{\gamma : [0,1] \to M \mid \gamma(0) = \phi(0), \gamma(1) = \psi(0)\}.\end{aligned}$$

The left hand side is the homotopy fiber

$$\begin{tikzcd}
\mathrm{Emb}_{/M}(\phi,\psi) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \arrow[d] \\
* \arrow[r] & \mathrm{Emb}(\mathbb{R}^n, M)
\end{tikzcd}$$

The top right is $O(n)$ and the bottom right is the frame bundle of the tangent bundle. But the homotopy fiber of the inclusion of the fiber into the frame bundle is, by the Puppe sequence, just ΩM. That's exactly the right hand side. This proves it for $k = 1$.

For general k, the evaluation at 0 maps to the unordered configuration space $\mathrm{Conf}_k(M)_{\Sigma_k}$. Apply the exact same argument as the $k = 1$ case. Hence

$$\mathcal{D}\mathsf{isk}_{n/M}^0 \simeq \coprod_{k\geq 0} \mathrm{Conf}_k(M)_{\Sigma_k}.$$

We want to modify our hypercover argument from earlier to tell us that

$$\operatorname*{colim}_{\mathsf{Disk}_{n/M}^{=k,\text{iso eq}}} (\mathbb{R}^n)^k_{\Sigma_k} \xrightarrow{\sim} \mathrm{Conf}_k(M)_{\Sigma_k}$$

is an equivalence. Notice that it is important that the k's on the left and the right are the same. But $(\mathbb{R}^n)^k \simeq *$ whence the colimit is equivalent to $B(\mathsf{Disk}_{n/M}^{=k,\text{iso eq}} \simeq \mathrm{Conf}_k(M)_{\Sigma_k} \simeq \mathcal{D}isk_{n/M}^{=\text{k, iso eq}}$. $\square$

This is a handy result which, since localizations are both final and initial, allows us to compute factorization homology along either category.

Lemma 94 (AF 3.21)**.** *With f, M, N as in last lecture, the map $\mathcal{D}\mathsf{isk}_f \to \mathcal{D}\mathsf{isk}_{/M}$ is final.*

Proof. We use Quillen's theorem A: show $B(\mathcal{D}\mathsf{isk}_f^{U/})^{U \hookrightarrow M} \simeq *$. Notice that we have

$$\mathcal{D}\mathsf{isk}^{\partial}_{k/N} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}_{n/M} \xrightarrow{\mathrm{Emb}_{/M}(U,-)} \mathsf{Spaces}$$

and as a general feature of unstraightening,

$$B(\mathcal{D}\mathsf{isk}_f^{U/}) = \operatorname*{colim}_{\mathcal{D}\mathsf{isk}^{\partial}_{k/N}} \mathrm{Emb}_{/M}(U, f^{-1}V)$$

The right hand side is equivalent to $*$ since

$$\operatorname*{colim}_{\mathcal{D}\mathsf{isk}^{\partial}_{k/N}} \mathrm{Emb}(U, f^{-1}V) \xrightarrow{\simeq} \mathrm{Emb}(U, M).$$

Okay, so since I've lost basically all of you, I'll simplify the proof and finish the proof in the case of $\mathrm{Emb}^{\mathrm{fr}}$. We need to show that

$$\operatorname*{colim}_{\mathcal{D}\mathsf{isk}^{\partial}_{k/N}} \mathrm{Emb}^{\mathrm{fr}}(U, f^{-1}V) \to \mathrm{Emb}^{\mathrm{fr}}(U, M).$$

Replace the index category with $\mathsf{Disk}^{\partial}_{\mathsf{k/N}}$ by the proposition above. We thus need to show that we have an equivalence

$$\operatorname*{colim}_{\mathsf{Disk}^{\partial}_{k/N}} \mathrm{Conf}_{\pi_0 U}(f^{-1}V) \hookrightarrow \text{open}\, \mathrm{Conf}_{\pi_0 U}(M).$$

Now we apply the hypercover argument. Set $|\pi_0 U| = K$. For points $\{x_1, \ldots, x_k\} \in \mathrm{Conf}_k(M)$, want to show $B(\mathsf{Disk}^{\partial}_{k/N})_{\{x_1,\ldots,x_k\}}) \simeq *$. First we check that it's nonempty. Notice under f these points get sent to $\{fx_1, \ldots, fx_k\} \in N^k$. We can choose k little disks containing these images, call it V. Take the inverse image of V and this inverse image will now contain the disk. The argument that the category is cofiltered goes as usual.

For the non-framed case, the argument is similar, which you can find the in the paper. $\square$

Sam: we're showing localization for handles of index 0. Can we do this for higher index handles? John: If you enhance by some stratification story, yes. Otherwise there's an issue about convergence of the Goodwillie-Weiss tower.

20. Factorization homology is a homology theory [11/06/2017]

John: Where is $\mathsf{Disk}_{n/M}$ better than the script version? Call them A and B. For A we can apply our hypercover lemma since A is a poset. B is better because we have a pushforward formula coming from the fact that $\mathsf{Disk}_f \to \mathcal{D}\mathsf{isk}_{n/M}$ is final. In particular B is what is known as a sifted ∞-category, which we will talk about later. This is not true about A.

Recall that we want to prove that homology theories for manifolds valued in $\mathcal{V}$ are equivalent to n-disk algebras in $\mathcal{V}$ by the evaluation on the point morphism. We will prove this using handle decompositions. Recall that a handle attachment of index q to W^n is a map

$$\begin{tikzcd}
\partial D^q \times \mathbb{R}^{n-q} \arrow[r, hook] \arrow[d, hook] & D^q \times \mathbb{R}^{n-q} \arrow[dd] \\
\partial W \arrow[d, hook] & \\
W \arrow[r, hook] & W \cup_{\partial D^q \times \mathbb{R}^q} D^q \times \mathbb{R}^{n-q}
\end{tikzcd}$$

Theorem 95 (Smale, after Morse)**.** *Every smooth manifold can be built from the empty n-manifold from a sequence of handle attachments.*

Proof. See, for instance Milnor's Morse theory (but you will have to fill in some details). It might be in Milnor's h-cobordism book. **Don't** look at Smale's paper on it. Or any of his papers, for that matter. $\square$

Let's now turn to the classification of homology theories.

Proof. Given any $F : \mathcal{M}\mathsf{fld}_n \to \mathcal{V}$ we have a map

$$\int_{-} F|_{\mathcal{D}\mathsf{isk}_n} \to F$$

which is the counit of the adjunction between restriction (evaluation at the point) and LKan (factorization homology). For F a homology theory we will prove that the counit is an equivalence by induction on a handle presentation.

As a warmup, consider the case of thickened spheres: $S^k \times \mathbb{R}^{n-k}$. We wish to prove that the map

$$\int_{S^k \times \mathbb{R}^{n-k}} F|_{\mathcal{D}\mathsf{isk}_n} \to F(S^k \times \mathbb{R}^{n-k})$$

is an equivalence by induction on k. The base case is that of $k = 0$. In this case, since F is symmetric monoidal, $F(S^0 \times \mathbb{R}^n) = F(\mathbb{R}^n) \otimes F(\mathbb{R}^n)$. Recall that in the case where $\mathcal{V}$ is Spaces the tensor product is nothing but the Cartesian product. Since factorization homology is symmetric monoidal (see the lemma below) the left hand side also splits as a tensor product. This proves the base case (it also obviously follows from the fact that the restriction of the Kan extension yields what you started with, since $S^0 \times \mathbb{R}^n$ is an n-disk).

We now turn to the inductive step (still for the thickened sphere). Assume

$$\int_{S^j \times \mathbb{R}^{n-j}} F|_{\mathcal{D}\mathsf{isk}_n} \xrightarrow{\sim} F(S^j \times \mathbb{R}^{n-j})$$

is an equivalence for $j < k$. We use that we can write $S^k \times \mathbb{R}^{n-k} = \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \cup_{S^{k-1}\times\mathbb{R}\times\mathbb{R}^{n-k}} \mathbb{R}^k_- \times \mathbb{R}^{n-k}$. Now apply our excision result

$$\int_{S^k\times\mathbb{R}^{n-k}} F|_{\mathcal{D}\mathsf{isk}_n} \simeq \int_{\mathbb{R}^k_+\times\mathbb{R}^{n-k}} F|_{\mathcal{D}\mathsf{isk}_n} \otimes_{S^{k-1}\times\mathbb{R}\times\mathbb{R}^{n-k}} \int_{\mathbb{R}^k_-\times\mathbb{R}^{n-k}} F|_{\mathcal{D}\mathsf{isk}_n}$$

and notice that

$$F(S^k \times \mathbb{R}^{n-k}) \simeq F(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \otimes_{F(S^{k-1}\times\mathbb{R}\times\mathbb{R}^{n-k})} F(\mathbb{R}^k_- \times \mathbb{R}^{n-k}).$$

Applying the induction hypothesis to what we are taking the tensor product over, and applying homotopy invariance of this tensor product (recall that it was defined as a homotopy colimit), we obtain the desired equivalence. This completes the proof for thickened spheres.

Now let's turn to a general sequence of handle attachments starting from the empty set. Inductively, we assume the counit map is an equivalence for $k-1$ and we show it for k. Look at the diagram for W above in our description of Smale's theorem. By the inductive hypothesis, for W with fewer than k-handles,

$$\int_W F|_{\mathcal{D}\mathsf{isk}_n} \xrightarrow{\sim} F(W)$$

and similarly for (a thickened version of) the manifold on the top left and the manifold on the top right. Now applying the $\otimes$-excision principle and the induction hypothesis concludes the proof. $\square$

Lemma 96. *For $A \in \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V})$, the factorization homology is symmetric monoidal if $\otimes$ in $\mathcal{V}$ preserves colimits.*

Proof. Consider the composite

$$\mathcal{D}\mathsf{isk}_{n/M\sqcup N} \xrightarrow{\sim} \mathcal{D}\mathsf{isk}_{n/M} \times \mathcal{D}\mathsf{isk}_{n/N} \xrightarrow{A\times A} \mathcal{V}\times\mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

Then since $\otimes$ preserves colimits

$$\begin{aligned}\int_{M\sqcup N} A &\simeq \operatorname{colim}\left(\mathcal{D}\mathsf{isk}_{n/M} \times \mathcal{D}\mathsf{isk}_{n/N} \xrightarrow{\otimes\circ A\times A} \mathcal{V}\right)\\ &\simeq \otimes \operatorname{colim}\left(\mathcal{D}\mathsf{isk}_{n/M} \times \mathcal{D}\mathsf{isk}_{n/N} \to \mathcal{V}\times\mathcal{V}\right)\\ &\simeq \int_M A \otimes \int_N A.\end{aligned}$$

$\square$

Lemma 97. *For Z an n-connective pointed space, the functor $\mathsf{Mfld}_n \to \mathsf{Spaces}$ sending $M \mapsto \mathrm{Maps}_c(-, Z)$ is a $\times$-homology theory.*

Remark 98. There is an equivalence of infinity-categories, for G any topological group,

$$\mathsf{Spaces}_{/BG} \simeq G-\mathsf{Spaces}.$$

More generally, there is an equivalence $\mathsf{LFib}_{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathsf{Spaces})$. But we can deduce it more directly—in more point-set terms the left hand side is considered as spaces that are fibrations over BG and the right hand side as G-spaces where the G-action is free. There is a map from the right to the left taking quotients, and the map from the left to the right is taking the fiber above the distinguished point over BG. If we omit the words fibration/free we replace quotient/fiber with homotopy quotient/homotopy fiber. The reason this is true is that taking homotopy fibers preserves homotopy colimits.[1] Suppose we are given a space $E \to BG$ with fiber F. There is a map $F \times_G EG \to E$ over BG. Hence we have a map of two homotopy fiber sequences, so the LES in homotopy groups for a fibration shows us that $\pi_* E \cong \pi_* F \times_G EG$. Hence every space over BG is the fiber of the quotient of the induced action of G. Similarly for the converse: if G acts on F then the fiber of $F_G \to BG$ is just F again.

21. Nonabelian Poincaré duality VI [11/08]

Nilay: could you quickly go over again how the pushforward formula implies $\otimes$-excision of factorization homology? John: Sure. Choose a map $M \to [-1,1]$ that exhibits the decomposition of M into M' and M'' (there's a diagram we drew before). The pushforward formula tells us that

$$\int_{[-1,1]} f_* A = \int_M A$$

and we used finality to pass from a colimit over Disk partial or over $[-1,1]$ to a colimit over Δ^{op} to reduce to the two-sided bar construction which is precisely the $\otimes$-excision.

We now return to the proof of nonabelian Poincaré duality. Recall the n-disk algebra $\Omega^n Z := \mathrm{Maps}_c(\mathbb{R}^n,)$, for Z an n-connective pointed space.

Proof. We just need to show that $\mathrm{Maps}_c(-, Z)$ is a $\times$-homology theory in Spaces and then apply our classification of $\otimes$-homology theories to conclude that it is given by factorization homology. In other words in suffices to prove the last lemma from last lecture. $\square$

Learn the 2-sided bar construction.

Proof. Choose a decomposition $M \cong M' \cup_{M_0 \times \mathbb{R}} M''$. We need to show that the natural map $\mathrm{Maps}_c(M', Z) \times \mathrm{Maps}_c(M'', Z))_{\mathrm{Maps}_c(M_0 \times \mathbb{R}, Z)} \to \mathrm{Maps}_c(M, Z)$. is an equivalence. The source is the quotient by the E_1-algebra, $\mathrm{Maps}_c(M_0 \times \mathbb{R}, Z)$, in Spaces where the structure is given by the monoidal map

$$\mathsf{Disk}_1 \xrightarrow{M_0 \times -} \mathcal{M}\mathsf{fld}_n \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Spaces}.$$

Observe that $M_0 \hookrightarrow M$ is proper, whence we obtain a restriction map $\mathrm{Maps}_c(M, Z) \to \mathrm{Maps}_c(M_0, Z)$. Observe that this restriction map is a Serre fibration with a connected base because Z is n-connective and M_0 is $(n-1)$-dimensional (every map in the base is homotopic to the constant map). Let's look at the fibers of this map.

$$\begin{tikzcd} \mathrm{Maps}_c((M, M_0), (Z, Z*)) \arrow[r] \arrow[d] & \mathrm{Maps}_c(M, Z) \arrow[d] \\ * \arrow[r, "M_0 \to * \to Z"] & \mathrm{Maps}_c(M_0, Z) \end{tikzcd}$$

However, this space of compactly supported maps of pairs is

$$\mathrm{Maps}_c((M, M_0), (Z, *)) \simeq \mathrm{Maps}_c(M \setminus M_0, Z).$$

Recall that this is the same argument as $\Omega^n Z \simeq \mathrm{Maps}_c(int(D^n), Z)$. Finally, notice that $M \setminus M_0 \cong M' \sqcup M''$ (shrinking collars).

[1]Check, for instance, that taking set-theoretic fibers preserves colimits along closed inclusions.

Since the bottom right of the square is a connected space we find

$$\mathrm{Maps}_c(M' \sqcup M'', Z)_{\Omega\,\mathrm{Maps}_c(M_0,z)} \xrightarrow{\sim} \mathrm{Maps}_c(M, Z)$$

from the equivalence of $\Omega K-\mathsf{Spaces}$ with $\mathsf{Spaces}_{/K}$ for K connected (for which we have $B \simeq B(\Omega K)$ and $G \simeq \Omega BG$). Now, since, $\Omega\,\mathrm{Maps}_c(M_0, Z) \simeq \mathrm{Maps}_c(M_0 \times \mathbb{R}, Z)$ and since compactly supported maps functor sends disjoint unions to products, we obtain the desired result. □

This completes the proof of nonabelian Poincaré duality.

Matt: how difficult is the hypercover lemma? John: I wouldn't think of it as too difficult. There's a proof in Jacob's appendix of hypercovers. A better place to start is Segal's Classifying Spaces paper, which has the essential idea. Then various glossed up versions are due to Dugger and Isaaksen.

Remark 99. Nonabelian Poincaré duality in this form is the best route to computing $H_*\,\mathrm{Maps}_c(M, Z)$ when M is a manifold. Of course, factorization homology looks a lot more complicated than this but on the plus side, it has a lot more handles attached.

Example 100. Let $Z = \Sigma^n K$ for K connected. We know

$$C_*\,\mathrm{Maps}_c(M, \Sigma^n K) \simeq \int_M C_*(\Omega^n \Sigma^n K) = \oplus_{i \geq 0} C_* \left(\mathrm{Conf}_i(M) \times_{\Sigma_i} K^{\wedge i}\right)$$

since the argument is a free n-disk algebra in chain complexes. How would you do this otherwise? Even Sam is allowed to contribute. This is a hard problem.

Sam: rational homotopy theory? John: hmm. . . okay, but that depends on simply-connectedness and characteristic zero.

However, this result predates factorization homology, c.f. McDuff, Segal, Boedigheimer.

Remark 101. Let's quickly look at why nonabelian Poincaré duality reduces to Poincaré duality. Let $Z = K(A, i)$, with $i \geq n$. Consider

$$\pi_j\,\mathrm{Maps}_c(M, K(A, i)) \cong [M, \Omega^j K(A, i)]_c \cong H_c^{i-j}(M; A),$$

so we have cohomology on the right hand side. On the other side we have

$$\int_M \Omega^n K(A, i).$$

This takes some arguing that we haven't done but I'll tell you the end result. There is the Dold-Kan functor followed by geometric realization $\mathsf{DK} : \mathsf{Ch} \to \mathsf{Spaces}$ sending $A[i] \mapsto K(A, i)$. Now recall that

$$\int_M V \simeq C_*(M, V)$$

for a chain complex V. Set V to be $A[i - n]$. Hence

$$\int_M \Omega^n K(A, i) \simeq |C_*(M, A[i - n])|.$$

Nilay: what happened to orientation? John: notice that these are n-disk algebras, and not E_n-algebras. In particular there is a nontrivial action of $O(n)$ on $K(A, i)$. The $\mathbb{Z}/2$ of the components of $O(n)$ is giving us the orientation twist. In particular, the left hand side yields the *twisted homology*.

22. Commutative algebras [11/10/2017]

Exercise 102. This is **Homework 11**, due next Friday. Suppose we have a functor $\mathcal{K} \to \mathsf{Opens}(X) \to \mathsf{Top}$ (of ordinary categories). When is the ordinary colimit isomorphic to X? More precisely, what is the weakest condition on $\mathcal{K}_x, x \in X$ to ensure an homeomorphism. Hint: the condition will be weaker than required the classifying space be contractible.

Definition 103. The **Hochschild homology** of a dga A is the derived tensor product $A \otimes_{A \otimes A^{\mathrm{op}}} A$ (the subscript tensor product is also derived). Here the $A \otimes A^{\mathrm{op}}$ module structure on A is given by the map from the derived tensor product to the usual tensor product.

Corollary 104. *Factorization homology of the circle is Hochschild homology*

$$\int_{S^1} A \simeq HH_* A.$$

Proof. We can write $S^1 = \mathbb{R}_+ \cup_{S^0 \times \mathbb{R}} \mathbb{R}_-$ so the excisiveness of factorization homology yields

$$\int_S^1 A \simeq \int_{\mathbb{R}_+} A \otimes_{\int_{S^0 \times \mathbb{R}} A} \int_{\mathbb{R}^-} A \simeq A \otimes_{A \otimes A^{\mathrm{op}}} A.$$

□

Example 105. Recall we had a functor $\mathcal{D}\mathsf{isk}_n \xrightarrow{\pi_0} \mathsf{Fin}$ sending a disjoint union of I copies of $\mathbb{R}^n$ to I. This yields amap

$$\mathsf{Alg}_{\mathrm{com}}(\mathcal{V}) \to \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V}).$$

Definition 106. We say that $\mathcal{C}$ is **tensored over** Spaces if for all $c \in \mathcal{C}$, $X \in \mathsf{Spaces}$, there exists $X \boxtimes c \in \mathcal{C}$ such that $\mathrm{Maps}_{\mathcal{C}}(X \boxtimes c, -) \simeq \mathrm{Maps}_{\mathcal{C}}(c, -)^X$. Here $Y^X := \mathrm{Maps}(X, Y)$.

Exercise 107. This is **homework 12**. Check that $X \boxtimes c \simeq \mathrm{colim}(X \to * \xrightarrow{c} \mathcal{C})$. On the right we regard X as an ∞-groupoid.

Theorem 108. *If $A \in \mathsf{Alg}_{com}(\mathcal{V})$ then*

$$\int_M A \simeq M \boxtimes A$$

where on the right we have a tensor of $A \in \mathsf{Alg}_{com}(\mathcal{V})$.

Proof. We induct on a handle presentation of M. Case 1 is when $M = \mathbb{R}^n$, in which case $\int_{\mathbb{R}^n} A \simeq A$. The right hand side is $\mathbb{R}^n \boxtimes A \simeq * \boxtimes A \simeq A$. Case 1' is when M is a disjoint union of $\mathbb{R}^n$'s, so on the left we have $A^{\otimes I}$. On the right we have $I \boxtimes A$. Notice that

$$\mathrm{Maps}(I \boxtimes A, -) \simeq \mathrm{Maps}(A, -)^I$$

shows that $I \boxtimes A$ is the coproduct. For instance, if we work in the case of chain complexes, we get $\otimes_I A$.

We want to show that

$$\mathsf{Mfld}_n \to \mathsf{Spaces} \xrightarrow{-\boxtimes A} \mathsf{Alg}_{\mathrm{com}}(\mathcal{V}) \to \mathcal{V}$$

is $\otimes$-excisive. Check $(M' \boxtimes A) \otimes_{M_0 \times \mathbb{R} \boxtimes A} (M'' \boxtimes A) \simeq M \boxtimes A$. In particular, it's enough to show by Yoneda,

$$\mathrm{Maps}_{\mathsf{Alg}_{\mathrm{com}}}((M' \boxtimes A) \otimes_{M_0 \times \mathbb{R} \boxtimes A} (M'' \boxtimes A), c) \simeq \mathrm{Maps}_{\mathsf{Alg}_{\mathrm{com}}}(A, c)^M.$$

We will take it for granted that $B \otimes_A B'$ is a pushout in the ∞-category $\mathsf{Alg}_{\text{com}}(\mathcal{V})$ (for the 1-categorical statement see the exercise below). Applying Maps we obtain a homotopy fiber product

$$\begin{aligned}
\mathrm{Maps}(M' \boxtimes A, c) \times_{\mathrm{Maps}(M_0 \times \mathbb{R} \boxtimes A, c)} \mathrm{Maps}(M'' \otimes A, c) &\simeq \mathrm{Maps}(A, c)^{M'} \times_{\mathrm{Maps}(A,c)^{M_0 \times \mathbb{R}}} \mathrm{Maps}(A, c)^{M''} \\
&\simeq \mathrm{Maps}(A, c)^{M' \sqcup_{M_0 \times \mathbb{R}} M''} \\
&\simeq \mathrm{Maps}(A, c)^m \simeq \mathrm{Maps}(A \boxtimes A, C),
\end{aligned}$$

so we are done. □

Exercise 109. This is **Homework 13**. Show that the usual pushout in the 1-category of commutative algebras is the two-sided tensor product you expect. Notice that this is a special feature of commutative algebras, not associative algebras in general.

The ∞-categorical formulation is more difficult. You prove it for coproducts and then show that the forgetful functor of commutative algebras in $\mathcal{V}$ to $\mathcal{V}$ preserves geometric realization.

Theorem 110. *Given Z n-connected,*

$$M \boxtimes C^* Z \simeq C^*(\mathrm{Maps}(M, Z)).$$

This makes sense because infinity-categorically, $\mathsf{Alg}_{com}(\mathsf{Ch}) \simeq \mathsf{Alg}_{\mathcal{E}_\infty}(\mathsf{Ch})$.

Remark 111. The condition on Z can be weakened considerably, but the statement above is good enough for our purposes.

Theorem 112 (Convergence of the Eilenberg-Moore spectral sequence)**.** *Given a homotopy pullback diagram of spaces satisfying some conditions (say Y simply connected for simplicity),*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

then the usual map

$$C^*(Y') \otimes_{C^*(Y)} C^*(X) \to C^*(X')$$

from the derived tensor product is an equivalence.

Proof. The statement is definitely true for when $M = *$. Now we have shown that

$$M' \sqcup_{M_0 \times \mathbb{R}} M'' \boxtimes C^* Z \simeq M' \boxtimes C^* Z \otimes_{M_0 \boxtimes C^* Z} M'' \boxtimes C^* Z$$

so we induct. The inductive hypothesis is that

$$M' \sqcup_{M_0 \times \mathbb{R}} M'' \boxtimes C^* Z \simeq C^* \mathrm{Maps}(M', Z) \otimes_{C^* \mathrm{Maps}(M_0, Z)} C^* \mathrm{Maps}(M'', Z)$$

Apply the spectral sequence theorem above for

$$X' = \mathrm{Maps}(M, Z) \quad Y' = \mathrm{Maps}(M', Z) \quad X = \mathrm{Maps}(M'', Z) \quad Y = \mathrm{Maps}(M_0, Z)$$

yields the desired result. □

Sean: is there a reason to use the factorization homology definition of Hochschild homology? John: well I never like using the old definition, because there's no mention of the circle, even though I don't know a result about Hochschild homology that I can't do without factorization homology.

No class next Friday.

23. Local systems and dualities [11/13/2017]

23.1. **Local systems.** Recall the statement about the convergence of the Eilenberg-Moore spectral sequence from last time. This spectral sequence is given under certain conditions. There is always a convergent spectral sequence given a diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & A' \cong B' \otimes_B A \end{tikzcd}$$

Fix these two Tor's

such that This is just a purely algebraic fact about dgas. Converting it into the first spectral sequence above is just given by the statement from last time.

For a proof sketch of the statement from last time, see lecture 23 from Lurie's class on the Sullivan conjecture at MIT. We will follow those notes. We say that a **local system of chain complexes** on Y is a functor $L : Y \to \mathsf{Ch}$ of ∞-categories. In terms of quasicategories, it is a map $\operatorname{Sing} Y \to \mathsf{Ch}$. We define the cohomology

$$C^*(Y, L) := \lim Y \xrightarrow{L} \mathsf{Ch}$$

For $L = A$ the constant local system, $C^*(Y, L)$ is the usual cohomology with coefficients in A because $C_*(X, L) = \operatorname{colim}_Y L$, and we can just take linear duals.

Observe now that given $f : X \to Y$ there is a local system $L_0 : Y \to \mathsf{Ch}$ that, heuristically, sends $y \in Y$ to $C^*(f^{-1}\{y\})$. Similarly for $g : Y' \to Y$ we get a localy system L_1. This yields

$$C^* L_0 \otimes_{C^* Y} C^* L_1 \to C^*(L_0 \otimes L_1) \simeq C^* X'$$

where we have used the Kunneth theorem (assume finite-dimensional cohomology groups in each degree). We say that L is **good**

(1) if $H^*(L(y)) = 0$ for $* < 0$ and all $y \in Y$,
(2) and if for any L' satisfying (1), the natural map

$$C^* L \otimes_{C^* Y} C^* L' \to C^*(L \otimes L')$$

is an equivalence.

Now, is *good*? Well, of course, this is not real life so we're good. For instance the zero local system is good because zero equals zero. So that's good. A less trivial example is the constant local system with value $\mathbb{Z}$, as is easy to check. The class of good local systems is closed under extensions, i.e. if we have

$$\begin{tikzcd} L_1 \arrow[r, hook] & L \arrow[d] \\ & L_2 \end{tikzcd}$$

such that $L_2 = \operatorname{coker}(L_1 \to L)$ then if L_1, L_2 are good then so is L. This is because the source and target of condition 2 preserve finite colimits.

If $\pi_1(Y)$ is nilpotent and if L is finite-dimensional concentrated in a single degree then it is a sequence of extensions by trivial $\pi_1 Y$-representations (this is some representation theory). Now induct on $L \to \varprojlim \tau^{\leq k} L$

$$\begin{tikzcd} \tau^{\leq k} L \arrow[d] & A[k] \arrow[l] \\ \tau^{\leq k-1} L & \end{tikzcd}$$

where we are using condition 1 to have a base case (really all we need is boundedness).

23.2. **Spooky duality.** Recall that we showed

$$\int_M C^* X \simeq M \boxtimes C^* X \simeq C^* \operatorname{Maps}(M, X)$$

where we interpret $C^* X$ as a commutative algegbra ∞-categorically, and the second equivalence assumes some conditions on X. This coincidentally looks familiar; we also have the statement

$$\int_M C_*(\Omega^n X) \simeq C_* \operatorname{Maps}_c(M, X)$$

for X pointed n-connective, which is NAPD. Observe that if M is a closed n-manifold, this implies that

$$\left(\int_M C_*(\Omega^n X)\right)^\vee \simeq \int_M C^* X.$$

This is not a coincidence! What we will talk about next is Koszul duality. It is a general feature of n-disk algebras which generalizes this relation here.

Let's recall how factorization homology works for manifolds with boundary. Given a functor $A : \mathcal{D}\mathsf{isk}_n^\partial \to \mathcal{V}$, we define

$$\int_M A \simeq \operatorname{colim} A$$

as usual. Now there are two fundamental objects of $\mathcal{V}$: $A(\mathbb{R}^n)$ and $A(\mathbb{R}^n_{\geq 0})$. Notice that NAPD also works for manifolds with boundary, where $\Omega^n X : \mathcal{D}\mathsf{isk}_n^\partial \to \mathsf{Spaces}$. The same proof works verbatim—not even *mutatis mutandis.*

Example 113. Consider

$$\int_{D^n} \Omega^n X \simeq \operatorname{Maps}_c(D^n, X) \simeq \operatorname{Maps}(D^n, X) \simeq X$$

where D^n is the closed N-disk. In other words, taking factorization homology over the closed n-disk is an n-fold delooping!

Notice, for future reference, that this implies

$$C^* X \simeq \left(\int_{D^n} C_* \Omega^n X\right)^\vee.$$

What happens if we do

$$\int_{D^n} C^* X = ?$$

Let's not get greedy—consider $n = 1$. Here the n-disk algebra sends $\mathbb{R}_{\geq 0} \mapsto \mathbb{Z}$ and $\mathbb{R} \mapsto A$ for some A. Picture putting the algebra A on the interior and $\mathbb{Z}$ on the boundary. But

$$\int_{D^1} C^* X \simeq \mathbb{Z} \otimes_{C^* X} \mathbb{Z} \simeq C^*(\Omega X)$$

from the proof of NAPD, where we had finality of $\Delta^{\mathrm{op}} \hookrightarrow \mathcal{D}\mathsf{isk}^{\partial,\mathrm{or}}_{1/[-1,1]}$. Using the pushforward formula for $D^n = (D^1)^n \to D^{n-1}$, we identify

$$\int_{D^n} C^* X \simeq \int_{D^1} \cdots \int_{D^1} C^* X \simeq C^*(\Omega^n X).$$

What if we do it *again*? Given some finiteness conditions, we see that

$$C_* \Omega^n X \simeq \left(\int_{D^n} C^* X \right)^{\vee}.$$

In light of this example, here's an idea: maybe there's a functor

$$\mathbb{D} : \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathrm{aug}}}(\mathcal{V}) \to \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathrm{aug}}}(\mathcal{V})$$

such that

$$\left(\int_M A \right)^{\vee} \simeq \int_M \mathbb{D} A$$

when M is closed.

Matt: what does dualizing mean in a general $\mathcal{V}$? John: what I've said will only apply when $\mathcal{V}$ is stable.

24. Koszul duality [11/15/2017]

Koszul duality is something not about algebras, but about augmented algebras.

Definition 114. We say that A is an augmented k-algebra, for k commutative, if we are given a map $A \to k$ of k-algebras.

In particular, the unit map composed with the augmentation $k \to A \to k$ is necessarily the identity map on k. If we now recall that a commutative algebra in $\mathcal{V}$ (symmetric monoidal) is equivalent to the data of a symmetric monoidal functor $\mathsf{Fin} \xrightarrow{A} \mathcal{V}$ sending $I \mapsto A^{\otimes I}$.

Definition 115. An **augmented commutative algebra in** $\mathcal{V}$ is a symmetric monoidal functor $\mathsf{Fin}_* \xrightarrow{A} \mathcal{V}$ (where the monoidal structure on finite pointed sets is, if you like, the wedge product).

If we think of the algebra as the value of the functor A on $\langle 1\rangle_*$, there is a map $A(\langle 1\rangle_*) \to A(*)$. This is exactly our classical augmentation. There is of course a functor from Fin to Fin_* so restriction along this functor is the forgetting of the augmentation data. This yields the following definition.

Definition 116. An **augmented n-disk algebra** is a symmetric monoidal functor

$$\mathcal{D}\mathsf{isk}_{n,*} \xrightarrow{A} \mathcal{V}.$$

Here the domain has objects $\coprod \mathbb{R}^n \sqcup *$, and morphisms, away from the components that they send to $*$, are embeddings. The symmetric monoidal structure is the wedge product.

Observe that an augmented commutative algebra yields an augmented n-disk algebra, by the usual restriction.

Notice that there is a symmetric monoidal functor

$$\mathcal{D}\mathsf{isk}_n^\partial \to \mathcal{D}\mathsf{isk}_{n,*}$$

sending

$$\coprod_I \mathbb{R}^n \sqcup \coprod_J \mathbb{R}^n_{\geq 0} \mapsto \coprod_I \mathbb{R}^n \sqcup *.$$

From this observation we find that there is a restriction functor $\mathsf{Alg}^{\mathrm{aug}}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V}) \to \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^\partial}(\mathcal{V})$, which yields

Corollary 117. *For A an augmented n-disk algebra, we can define*

$$\int_M A = \operatorname*{colim}_{\mathcal{D}\mathsf{isk}_n^\partial/M} A$$

for M a manifold with boundary.

One might say that n-disk boundary algebras are more general, so why bother restricting when working with manifolds with boundary? The reason is because Koszul duality applies to the augmented algebras.

Theorem 118. *There exists a contravariant functor*

$$\mathsf{Alg}^{aug}_{\mathcal{D}\mathsf{isk}_n}(\mathsf{Ch})^{op} \longrightarrow \mathsf{Alg}^{aug}_{\mathcal{D}\mathsf{isk}_n}(\mathsf{Ch})$$

that on underlying objects assigns

$$A \mapsto \left(\int_{D^n} A\right)^\vee.$$

Here we are stating the theorem in minimal generality—what we need is that $\mathcal{V}$ is stable.

Last time we saw that

$$\left(\int_{D^n} C_*(\Omega^n X)\right)^\vee \simeq (C_* \operatorname{Maps}_c(D^n, X))^\vee \simeq C^* X.$$

Here $C_* X$ is of course augmented. Given some (finiteness, etc.) conditions we also saw that

$$\left(\int_{D^n} C^* X\right)^\vee \simeq (C^* \Omega^n X)^\vee \simeq C_* \Omega^n X.$$

Exercise 119. Recall that if M is without boundary then we had

$$\int_M \mathsf{Free}_n(V) \simeq \coprod_{i \geq 0} \operatorname{Conf}_i(M) \times_{\Sigma_i} V^i$$

It is easy to see that the free n-disk algebra on the space V is augmented. Similarly for $V \in (\mathcal{V}, \otimes)$,

$$\int_M \mathsf{Free}_n(V) \simeq \coprod_{i \geq 0} \operatorname{Conf}_i(M) \boxtimes_{\Sigma_i} V^{\otimes i}.$$

For **homework 14**, allow M to have a boundary and let $\mathcal{V} = \mathsf{Ch}$. Show that

$$\int_M \mathsf{Free}_n(V) \simeq \bigoplus_{i \geq 0} \operatorname{Conf}_i(M)/\partial \boxtimes_{\Sigma_i} V^{\otimes i}.$$

Here $\operatorname{Conf}_i(M)/\partial$ is the subspace of $\operatorname{Conf}_i(M)$ such that at least one point lies in ∂M. Note also that $\boxtimes$ is the tensor with pointed spaces. Then show that for $M = D^n$,

$$\int_{D^n} \mathsf{Free}_n(V) \simeq \mathbb{Z} \oplus V[n].$$

Let me give you some hints. First note that the tensor $X \boxtimes V \simeq \tilde{C}_*(X) \otimes V$ is reduced chains. For $i = 0$ we get $* \boxtimes_{\Sigma_0} V^{\otimes 0} = \mathbb{Z}$. For $i = 1$ we have

$$\operatorname{Conf}_1(D^n)/\partial \boxtimes_{\Sigma_1} V \simeq D^n/\partial D^n \boxtimes V \simeq V[n]$$

since $\tilde{C}_*(S^n) \otimes V = \mathbb{Z}[n]$. What I've left for you for homework is to show that $\operatorname{Conf}_i(D^n)/\partial \simeq *$.

For future reference, what is the n-disk algebra structure on the dual of this? We have

$$\left(\int_{D^n} \mathsf{Free}_n(V)\right)^\vee \simeq \mathbb{Z} \oplus V^\vee[-n].$$

This is a general feature of Koszul duality—the dual of a free algebra is a trivial algebra.

Add picture here

25. The cotangent functor [11/20/2017]

Let's think about one of the past homework problems together.

Lemma 120. *We have that* $\mathrm{Conf}_i(D^n)/\partial \simeq *$ *for* $i \geq 2$.

Proof. Note that for N^{n-1} with $\partial N = \varnothing$ then

$$\partial \,\mathrm{Conf}_k\,(N \times [-1,0)) \to \mathrm{Conf}_k(N \times [-1,0))$$

is a homotopy equivalence. We can do this by exhibiting a deformation retraction of this map. Let me sketch how this works for the circle (see sketch in notebook). Notice that this argument does not work for $k = 0$. In that case the right hand side is a point and the left hand side is the empty set and there are no maps from the point to the empty set. Now notice that by translating the first point to the center $\mathrm{Conf}_i(D^n) \simeq \mathrm{Conf}_{i-1}(D^n - \{0\})$ (can show this by replacing D^n with $\mathbb{R}^n$ and noting that $\mathrm{Conf}_i(\mathbb{R}^n) \to \mathbb{R}^n$ has the right fiber). □

Okay, back to Koszul duality. It is in fact more general than just the form it takes for n-disk algebras. Recall that the way we stated it we had a contravariant functor $(\int_{D^n} -)^\vee : \mathsf{Alg}_n^{\mathrm{aug,op}} \to \mathsf{Alg}_n^{\mathrm{aug}}$ (abbreviating our notation of n-disk algebas). For each n Koszul duality looks symmetric but in the limit where $n \to \infty$, the symmetry is broken:

$$\begin{tikzcd} \mathsf{Alg}_{\mathrm{comm}} \arrow[r, bend left, "T"] & \mathsf{Alg}_{\mathrm{Lie}} \arrow[l, bend left, "C^*_{\mathrm{Lie}}"] \end{tikzcd}$$

$$\mathsf{Alg}_n^{\mathrm{op}} \longrightarrow \mathsf{Alg}_n$$

$$\mathsf{Alg}_{n-1}^{\mathrm{op}} \longrightarrow \mathsf{Alg}_{n-1}$$

Let's explain the maps at the very top, i.e. standard Koszul duality, in two versions. In version 1, suppose we have some algebraic structure $\mathcal{O}$ such that there is a trivial algebra functor

$$\mathsf{Ch} \xrightarrow{\mathbb{Z}\oplus -} \mathsf{Alg}_{\mathcal{O}}(\mathsf{Ch}).$$

This functor always preserves limits because it's a right adjoint. More explicitly given $J \xrightarrow{A} \mathsf{Alg}_{\mathcal{O}}(\mathsf{Ch}) \xrightarrow{\mathsf{fgt}} \mathsf{Ch}$ one shows that the limit of this composite functor has an algebra structure. Have

$$\begin{tikzcd} \lim \mathsf{fgt} \circ A \otimes \lim \mathsf{fgt} \circ A \arrow[rr, dashed] \arrow[dr] & & \lim \mathsf{fgt} \circ A \\ & \lim\left(J \times J \xrightarrow{(\mathsf{fgt}\circ A)\otimes(\mathsf{fgt}\circ A)} \mathsf{Ch}\right) \arrow[ur] & \end{tikzcd}$$

which gives us what we want. This is all to say that there exists a left adjoint from $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Ch}) \to \mathsf{Ch}$, call it L for the "cotangent space" at the augmentation. Hence we see that $L \circ \mathsf{Free}_{\mathcal{O}}$ is left adjoint to $\mathsf{fgt} \circ \mathsf{triv} = \mathrm{id}$, which implies that $L \circ \mathsf{Free}_{\mathcal{O}} \simeq \mathrm{id}$. Observe now that L sends free algebras to what they are free on $L(\mathsf{Free}_{\mathcal{O}} V) = V$. Moreover L preserves colimits. Geometrically if you think of $\mathrm{Spec}\, A$ then the augmentation is a point $A \to \mathbb{Z}$ and L is the cotangent space encoding infinitesimal data near the point. Big diagram here

If $\mathcal{O}$ is the operad for commutative algebras then $\mathsf{Free}_{\mathcal{O}}(V) = \mathrm{Sym}(V) = \mathcal{O}(\mathbb{A}_V)$ that is naturally augmented to $\mathbb{Z}$. Then $L(\mathrm{Sym}\, V) = V$ is the usual Zariski cotangent space at 0.

Theorem 121. *For A an n-disk algebra,*

$$\mathbb{Z} \oplus LA[n] \simeq \int_{D^n} A.$$

I'll maybe only sketch this proof (the details are in my Ph.D. thesis).

Proof sketch. We have already calculated that for $A = \mathsf{Free}_n(V)$,

$$\int_{D^n} \mathsf{Free}_n(V) \simeq \mathbb{Z} \oplus V[n],$$

which followed from the fact that $\mathrm{Conf}_i(D^n)/\partial \simeq *$ for $i \geq 2$. On the other hand, $L\mathsf{Free}_n(V) = V$ so

$$\int_{D^n} A \simeq \mathbb{Z} \oplus LA[n].$$

for A free. Thus the two agree on free algebras.

The next step is to show that

$$\int_{D^n} - : \mathsf{Alg}_n^{\mathrm{aug}}(\mathsf{Ch}) \to \mathsf{Ch}$$

preserves homotopy colimits of Δ^{op}-diagrams, i.e. for any

$$\Delta^{\mathrm{op}} \xrightarrow{A_\bullet} \mathsf{Alg}_n^{\mathrm{aug}} \xrightarrow{\int_{D^n}} \mathsf{Ch}$$

there is always a map $|\int_{D^n} A_\bullet| \to \int_{D^n} |A_\bullet|$ and this map is an equivalence. This statement is a consequence of the fact that all reasonable functors (those appearing above such as fgt and Free) preserve geometric realization. We have the iterated free forget simplicial resolution (monadic?) for A and this computes $|\mathsf{Free}_n^\bullet A|$ which tells us that

$$\int_{D^n} A \simeq \int_{D^n} |\mathsf{Free}_n^\bullet A| \simeq |\int_{D^n} \mathsf{Free}_n^\bullet A| \simeq |\mathbb{Z} \oplus L\mathsf{Free}_n^\bullet A[n]| \simeq \mathbb{Z} \oplus LA[n].$$

$\square$

Since we have only a bit of time left let me give some intuition for why the geometric realization is preserved.

Lemma 122. *The functor $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$ is final (i.e. Δ^{op} is "sifted").*

We don't have time to prove this. Now consider the composite functor $\Delta^{\mathrm{op}} \xrightarrow{A^\bullet} \mathsf{Alg} \xrightarrow{\mathsf{fgt}} \mathsf{Ch}$ which we denote by $A_\bullet$. We produce a multiplication on $|A_\bullet|$. We have

$$\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{A \times A} \mathsf{Ch} \times \mathsf{Ch} \xrightarrow{\otimes} \mathsf{Ch}$$

sending n to $A_n \otimes A_n$. Now $\mathrm{colim}(\Delta^{\mathrm{op}} \to \mathsf{Ch}) = |A \otimes A| \to |A_\bullet|$ By finality this is equivalent to $\mathrm{colim}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{A \otimes A} \mathsf{Ch}$. But this map goes through $\mathsf{Ch} \times \mathsf{Ch}$ via tensor product. Since tensor product preserves colimits this is equivalent to $\mathrm{colim}(\Delta^{\mathrm{op}} \xrightarrow{\Delta} \mathsf{Ch}) \otimes \mathrm{colim}\, A$.

Grisha: can you do something similar for more general things like monads? John: it's not true in that level of generality as can be seen by the if and only if statement in the Bar-Beck theorem.

26. Koszul duality II [11/27/2017]

Observe the following. For $\mathcal{K}$ a sifted category (nonempty with the diagonal functor final), given a diagram $\mathcal{K} \xrightarrow{A} \mathsf{Alg}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathcal{V}$. Then $\operatorname{colim}(\mathcal{K} \xrightarrow{A} \mathcal{V})$ is an algebra. This is because we have the composite

$$\mathcal{K} \to \mathcal{K} \times \mathcal{K} \xrightarrow{A \times A} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

and a natural transformation (via multiplication) from this composite to the functor A. Hence we obtain a map

$$\operatorname{colim}(\mathcal{K} \to \mathcal{K} \times \mathcal{K} \to \mathcal{V}) \to \operatorname{colim}(\mathcal{K} \xrightarrow{A} \mathcal{V}).$$

By finality and the fact that the $\otimes$ in $\mathcal{V}$ preserves colimits, we obtain

$$\operatorname{colim}(\mathcal{K} \times \mathcal{K} \to \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}) \simeq \otimes \operatorname{colim}(\mathcal{K} \times \mathcal{K} \to \mathcal{V} \times \mathcal{V}) \simeq \operatorname{colim}(\mathcal{K} \to \mathcal{V})^{\otimes 2}.$$

Hence siftedness is important for preserving algebraic structures.

The following more general result can be found in Lurie's Higher Algebra, though it is an older result.

Theorem 123. *The forgetful functor* $\mathsf{fgt} : \mathsf{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes}) \to \mathcal{V}^{\otimes}$ *preserves sifted colimits if* $\otimes$ *distributes over sifted colimits.*

Lemma 124. *The category* Δ^{op} *is sifted.*

We will need to recall how subdivision works for simplicial sets. Write $\mathsf{nd}[p]$ or the nondegenerate simplices of $[p]$, which is the nerve of the poset ordered by inclusion. Clearly $\mathsf{nd}[0] = [0]$. One checks that

$$\mathsf{nd}[1] = 0 \to 01 \leftarrow 1.$$

For $\mathsf{nd}[2]$ one obtains the usual picture of the barycentric subdivision of the two-simplex. A little bit of work shows that we obtain a functor $\mathsf{nd} : \Delta \to \mathsf{sSet}$. We now define the barycentric subdivision as the left Kan extension of nd:

$$\begin{tikzcd} \Delta \arrow[r, "\mathsf{nd}"] \arrow[d, hook] & \mathsf{sSet} \\ \mathsf{sSet} \arrow[ur, dashed] & \end{tikzcd}$$

In other words,

$$\mathsf{sd}X := \operatorname*{colim}_{\Delta} \mathsf{nd}.$$

Lemma 125. *For X a simplicial set,* $|X| \cong |\mathsf{sd}X|$.

Proof sketch. I'll leave the details for you, but the first case is when $X = \Delta[p]$. Then check that $|\Delta[p]| = \Delta^p \cong |\mathsf{nd}[p]|$. More generally, reduce $|X|$ to each cell and apply the above logic. ☐

Lemma 126. *The diagonal functor* $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ *is final.*

Proof. We use Quillen's theorem A and show that for any $[p], [q]$, the classifying space

$$B\left(\Delta^{\mathrm{op}} \times_{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})^{[p] \times [q]/}\right) \simeq *$$

is contractible. This is a bit clunky but we can rewrite as showing

$$B\left(\Delta_{/[p]} \times_{\Delta} \Delta_{/[q]}\right) =: B(\Delta_{/\Delta[p] \times \Delta[q]})$$

by using the fact that B doesn't care about taking opposites. Notice that

$$\Delta^{\mathrm{inj}}_{/\Delta[p]\times\Delta[q]} \hookrightarrow \Delta_{/\Delta[p]\times\Delta[q]}$$

where the left is the subset of $T - [p], T - [q]$ such that $T - [p] \times [q]$ is injective (for T an element of the right), has an adjoint. This adjoint is

$$\Delta^{\mathrm{inj}}_{/\Delta[p]\times\Delta[q]} \leftarrow \Delta_{/\Delta[p]\times\Delta[q]}$$

Fix this

given by Hence

$$B(\Delta^{\mathrm{inj}}_{\Delta[p]\times\Delta[q]} = B(\Delta_{/\Delta[p]\times\Delta[q]}.$$

But the left hand side is precisely $|\mathsf{sd}\Delta[p] \times \Delta[q]| \simeq *$. $\square$

So far we've described what the factorization homology over the n-disk has to do with the cotangent space, but lets return to the general story.

Theorem 127 (Ayala, Francis - Poincaré/Koszul duality)**.** *For k a field, consider $A \in \mathsf{Alg}^{\mathrm{aug}}_{\mathscr{D}\mathsf{isk}_n}(\mathsf{Ch}_k)$ such that $\bar{A} = \ker(A \to k)$ is connected (i.e. $H_*\bar{A} = 0$ for $* \leq 0$) and $H_k A$ is finite-dimensional. Write $\mathbb{D}^n A = \int_{D^n} A)^\vee$. Then for M a closed n-manifold, there is an equivalence*

$$\left(\int_M A\right)^\vee \simeq \int_M \mathbb{D}^n A.$$

We have seen this already when A is an $\mathcal{E}_n$-enveloping algebra of a Lie algebra or is chains on a n-fold loop spaces. The idea for the proof in general is to filter both sides and show that the layers are the same. We will filter $\int_M A$ using Goodwillie functor calculus and we will filter $\int_M \mathbb{D}^n A$ using Goodwilie-Weiss manifold calculus. In particular, one of the surprises in this context is that Goodwillie functor calculus, here, is Koszul dual to Goodwillie-Weiss manifold calculus. Indeed, we spoke to people at conferences who worked with both subjects, and they were pretty surprised, so hopefully this isn't obvious to you.

26.1. **Goodwillie calculus.**

Definition 128. A functor $A : \mathsf{Alg}^{\mathrm{aug}}_{\mathscr{D}\mathsf{isk}^{\mathrm{fr}}_n} \to \mathsf{Ch}_k$ is **i-homogeneous** if there exist $F_i \in \mathsf{Ch}_k$ equipped with actions of Σ_i on F_i such that $A \mapsto F_i \otimes_{\Sigma_i} (LA)^{\otimes i}$.

Definition 129. We say that F is **n-excisive** if it belongs to the closure of ℓ-homogeneous functors for $\ell \leq n$ under all limits in $\mathrm{Fun}(\mathsf{Alg}^{\mathrm{aug}}_{\mathscr{D}\mathsf{isk}^{\mathrm{fr}}_n}, \mathsf{Ch}_k)$.

Definition 130. The **n-excisive approximation** $P_n F$ to F is the universal n-excisive functor with a natural transformation $F \to P_n F$.

Lemma 131. *Write, for $A \in \mathsf{Alg}_{\mathcal{E}_n}$,*

$$\left(P_k \int_M\right)(A) =: P_k \int_M A.$$

There is a tower

$$P_k \int_M A \to P_{k-1} \int_M A \to \cdots \to P_1 \int_M A$$

such that the fiber is given

$$\begin{tikzcd}
C_*(\mathrm{Conf}_k(M)) \otimes_{\Sigma_k} (LA)^{\otimes k} \arrow[r] \arrow[d] & P_k \int_M A \arrow[d] \\
k \arrow[r] & P_{k-1} \int_M A
\end{tikzcd}$$

27. Manifold calculus [11/29/2017]

27.1. **Goodwillie-Weiss manifold calculus.** Last time we gave the quickest introduction to Goodwillie calculus known to mankind. Let's now look at another "filtration", the cardinality filtration.

Definition 132. Consider the category $\mathcal{D}\mathsf{isk}^{\mathrm{nu}}_{n/M}$ which has objects n-disks embedding into M where the embeddings are surjections on π_0. The morphisms are the π_0-surjective embeddings. The "nu" stands for nonunital, as algebras over this will no longer have units. This category is a (not full) ∞-subcategory of $\mathcal{D}\mathsf{isk}_{n/M}$.

For the following it is important that our categories are ∞-categories and not just posets.

Lemma 133. *The map $\mathcal{D}\mathsf{isk}^{nu}_{n/M} \to \mathcal{D}\mathsf{isk}_{n/M}$ is final.*

Proof. Left as an exercise. □

Define $\mathcal{D}\mathsf{isk}^{\mathrm{nu},\leq i}_{n/M} \subset \mathcal{D}\mathsf{isk}^{\mathrm{nu}}_{n/M}$ to be the subcategory of n-disks that have $|I| \leq i$ components. Then we define

$$\tau^{\leq i} \int_M A = \operatorname{colim}\left(\mathcal{D}\mathsf{isk}^{\mathrm{nu},\leq i}_{n/M} \to \mathsf{Ch}_k\right).$$

Observe that since we can write $\mathcal{D}\mathsf{isk}^{\mathrm{nu}}_{n/M} = \varinjlim \mathcal{D}\mathsf{isk}^{\mathrm{nu},\leq i}_{n/M}$ as a sequential colimit and because colimits over colimits of categories are just iterated colimits (do this abstract nonsense yourself, and note that it's not true for limits), we find that

$$\int_M A = \varinjlim_i \tau^{\leq i} \int_M A$$

via the above lemma.

What are the layers in this filtration?

Lemma 134. *For M closed, the following is a (homotopy) pushout:*

$$\begin{tikzcd}
\tau^{\leq i-1} \int_M A \arrow[r] \arrow[d] & \tau^{\leq i} \int_M A \arrow[d] \\
k \arrow[r] & \mathrm{Conf}_i(M)^+ \otimes_{\Sigma_i} A^{\otimes i}
\end{tikzcd}$$

Here we are taking the one-point compactification of the configuration space. If you like, the bottom right can be written as twisted chains $\tilde{C}_*(\mathrm{Conf}_i(M)^+_{\Sigma_i}, A^{\otimes i})$

Proof idea. We have already shown that $\mathcal{D}\mathsf{isk}^{\mathrm{nu},=i}_{n/M} \simeq \mathrm{Conf}_i(M)_{\Sigma_i}$. So what we want to study is the quotient

$$\mathcal{D}\mathsf{isk}^{\mathrm{nu},\leq i}_{n/M} / \mathcal{D}\mathsf{isk}^{\mathrm{nu},\leq i-1}_{n/M} \simeq \mathrm{Conf}_i(M)^+_{\Sigma_i}$$

and then apply some formal nonsense. □

Proposition 135. *For $k \oplus V$ a trivial $\mathcal{E}_n$-algebra the cardinality filtration splits as*

$$\int_M k \oplus V \simeq \bigoplus_{i \geq 0} \mathrm{Conf}_i(M)^+ \otimes_{\Sigma_i} V^{\otimes i}.$$

Compare this to the case of Goodwillie calculus for free algebras where we have a splitting

$$\int_M \mathsf{Free}_{\mathcal{E}_n}(W) \simeq \bigoplus_{i\geq 0} \mathrm{Conf}_i(M) \otimes_{\Sigma_i} W^{\otimes i}$$

which is quite similar. By Koszul duality,

$$\mathbb{D}^n(\mathsf{Free}_{\mathcal{E}_n} W) \simeq \left(\int_{D^n} \mathsf{Free}_{\mathcal{E}_n} W\right)^\vee \simeq (k \oplus W[n])^\vee \simeq k \oplus W^\vee[-n].$$

Proposition 136. *For A with connected augmentation ideal and H_*A finite rank, we have that*

$$\left(\int_M A\right)^\vee \simeq \int_M \mathbb{D}^n A.$$

Proof. We will play this proof slightly fast and loose, for lack of time. Suppose $A = \mathsf{Free}_{\mathcal{E}_n} V$. Then on the left we have

$$\left(\int_M \mathsf{Free}_{\mathcal{E}_n} V\right)^\vee = \left(\bigoplus \mathrm{Conf}_i(M) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee = \prod_{i\geq 0} \left(\mathrm{Conf}_i(M) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee.$$

Observe now the following important fact. We always have a map $\oplus_{i\geq 0} B_i \to \prod_{i\geq 0} B_i$. This is an equivalence if B_i is (for simplicity) i-connective as $i \to \infty$ (since finite sums and finite products are the same). Recall that V is 1-connective, whence $V^{\otimes i}$ is i-connective.[2] Hence $\mathrm{Conf}_i(M)^{\otimes} V^{\otimes i}$ is i-connective. Quotienting by Σ_i preserves this, as colimits preserve connectivity because we have an adjunction

$$\begin{tikzcd} \mathsf{Ch}_k^{\geq i} \arrow[r, bend left, "left"] & \mathsf{Ch}_k. \arrow[l, bend left, "\tau^{\geq i}"] \end{tikzcd}$$

Now the observation above allows us to simplify, up to equivalence, the direct product to a direct sum:

$$\bigoplus_{i\geq 0} \left(\mathrm{Conf}_i(M) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee.$$

We want this to be the same as

$$\int_M \mathbb{D}^n(\mathsf{Free}_{\mathcal{E}_n} V) \simeq \int_M k \oplus V^\vee[-n] \simeq \bigoplus_{i\geq 0} \mathrm{Conf}_i(M)^+ \otimes_{\Sigma_i} (V^\vee[-n])^{\otimes i}$$

In particular we want

$$\left(\mathrm{Conf}_i(M) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee \simeq \mathrm{Conf}_i(M)^+ \otimes_{\Sigma_i} (V^\vee[-n])^{\otimes i}.$$

Let's play with the left. The dual of a colimit is a limit so we have

$$\begin{aligned} \left(\mathrm{Conf}_i(M) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee &\simeq \left(C_*(\mathrm{Conf}_i(M)) \otimes_{\Sigma_i} V^{\otimes i}\right)^\vee \\ &\simeq \left(C_*(\mathrm{Conf}_i(M))^\vee \otimes (V^{\otimes} i)^\vee\right)^{\Sigma_i} \\ &\simeq C_*(\mathrm{Conf}_i(M))^\vee \otimes^{\Sigma_i} (V^{\otimes i})^\vee \\ &\simeq C^*(\mathrm{Conf}_i(M)) \otimes^{\Sigma_i} (V^\vee)^{\otimes i} \end{aligned}$$

[2]Recall that the definition between connected and connective is just a shift.

using the finiteness data that we have. On the right we have, playing a little loose (due to local coefficient issues)

$$\begin{aligned}\tilde{C}_*(\mathrm{Conf}_i(M)^+) \otimes_{\Sigma_i} (V^\vee[-n])^{\otimes i} &\sim \tilde{C}_* \mathrm{Conf}_i(M)^+ \otimes_{\Sigma_i} (V^\vee)^{\otimes i} [-ni] \\ &\sim \tilde{C}_* \mathrm{Conf}_i(M)^+ \otimes^{\Sigma_i} (V^\vee)^{\otimes i} [-ni]\end{aligned}$$

using the norm map being an equivalence (since $\mathrm{Conf}_i(M)$ is a finite CW complex on which the symmetric group acts freely). Finally we apply usual Poincaré duality, to obtain the left. This proves it for the free algebra.

For details, see AF's paper Poincaré/Koszul duality. □

Grisha: is it obvious that the Goodwillie tower converges here? John: we need the connectedness of the augmentation ideal. Otherwise the result is false. There is a more general result which is true, where Koszul dual of an algebra is a formal moduli problem. There is a form of factorization homology that takes as input a formal moduli problem. In this case all the conditions may be removed… notice also that here it was vital that we worked over a field so that we might swap connectivity and coconnectivity.

Appendix A. Exercises

Exercise 137. Construct homotopy equivalences

$$\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathrm{Diff}(\mathbb{R}^n) \simeq GL(n) \simeq O(n).$$

Exercise 138. Show that homotopy pullbacks are homotopy invariant.

Exercise 139. Show that compactly supported maps are covariant along open inclusions; in particular given an open inclusion $U \hookrightarrow V$, the induced map of spaces $\mathrm{Maps}_c(U, Z) \to \mathrm{Maps}_c(V, Z)$ is continuous. Here the topology on the mapping spaces is inherited from $\mathrm{Maps}_*(U^+, Z)$.

Exercise 140. Fill in the details of the proof that hocolim is homotopy invariant.

Exercise 141. Prove that there is a homeomorphism $|\Delta[n]| \cong \Delta^n$. Moreover, show that the geometric realization $|X|$ of a simplicial set X has the structure of a CW complex with an n-cell for each nondegenerate n-simplex.

Exercise 142. Show $\Delta[n] \star \Delta[m] \cong \Delta[n+m+1]$ and the corresponding statement for horns.

Exercise 143. Fill in the proof of Proposition 1.2.4.3 of HTT. In particular, show that ϕ and ψ are indeed inverse.

Exercise 144. Prove that a localization $\mathcal{C} \to \mathcal{D}$ of ∞-categories are both initial and final.

Exercise 145. Show, for functors between ordinary categories, that a left Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along a functor $g : \mathcal{C} \to \mathcal{D}$ is given by the formula

$$g_! F(d) = \operatorname*{colim}_{\mathcal{C}/d} F.$$

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.

[Hir94] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.