A BRIEF INTRODUCTION TO FACTORIZATION HOMOLOGY

NILAY KUMAR

Today we will motivate and give examples of a natural homology theory for manifolds known as factorization homology. We will try to keep the discussion as topological and geometrical as possible, glossing over some homotopical machinery and being imprecise in certain places. For instance all categories should probably be ∞-categories, but we will think of them as simplicial categories (the relevant enrichments will generally be coming from the compact-open and smooth compact-open topologies unless otherwise mentioned). Moreover in some places diagrams commuting should really be diagrams commuting up to homotopy. Our references are Ayala-Francis and the notes from John's class in the fall (available online on my website).

We start by recalling ordinary homology for spaces.

Definition 1. We say that a symmetric monoidal functor $\mathcal{F} : \mathsf{Spaces}^{\mathrm{fin}} \to \mathsf{Ch}$ is a *homology theory for spaces* if it satisfies excision, i.e. for any diagram of cofibrations

$$X' \hookleftarrow X \hookrightarrow X''$$

the resulting map of chain complexes

$$\mathcal{F}(X') \oplus_{\mathcal{F}(X)} \mathcal{F}(X'') \to \mathcal{F}(X' \sqcup_X X'')$$

is a quasi-isomorphism. Denote the category of homology theories by $H(\mathsf{Spaces}, \mathsf{Ch})$.

Here the (simplicial) category of spaces is symmetric monoidal under the disjoint union and the (simplicial[1]) category of chain complexes is symmetric monoidal under the direct sum. It is a classical result of Eilenberg and Steenrod that there is only one ordinary homology theory for each choice of coefficient object.

Theorem 2 (Eilenberg-Steenrod)**.** *There is an equivalence between homology theories for spaces (in chain complexes) and chain complexes*

$$\mathrm{ev}_* \colon H(\mathsf{Spaces}, \mathsf{Ch}) \rightleftarrows \mathsf{Ch} \colon C_*(\cdot\,; -)$$

where the equivalence is implemented by evaluation at the point and ordinary singular homology.

Remark 3. Here's an idea as to why this is believable. Probably one of you can take these rough notions and promote this to an actual proof. Suppose we have a chain complex A. Suppose $\mathcal{F}$ is a homology theory such that $\mathcal{F}(*) = A$. What is $\mathcal{F}(X)$ for $X \in \mathsf{Spaces}^{\mathrm{fin}}$? Recall that any space can be written as the homotopy colimit of its points

$$X = \mathrm{hocolim}_{\Pi_{\leq 1} X} *.$$

[1]The category Ch is naturally simplicially enriched via the presence of the cosimplicial object $C_*(\Delta^\bullet; \mathbb{Z})$.

Applying $\mathcal{F}$ and the excision property we obtain

$$\mathcal{F}(X) \simeq \mathrm{hocolim}_{\Pi^{\leq 1}X} A.$$

Recall that a homotopy colimit can be computed as a geometric realization (an ordinary colimit) of a simplicial object. Write the simplicial chain complex

$$\mathrm{srep}_n(\mathcal{F}(X)) = \coprod_{x_0 \to \cdots \to x_n} A_{x_n}$$

where A_{x_n} is a copy of A labelled by the point $x_n \in X$. The geometric realization is now

$$\mathcal{F}(X) \simeq \coprod_n (\mathrm{srep}_n(\mathcal{F}(X)) \otimes C_*(\Delta^n; \mathbb{Z})) \,/ \sim$$

where we have used the simplicial enrichment of Ch. Intuitively speaking, we are gluing copies of singular chains on simplices with coefficients in A according to the structure of X. This should yield $\mathcal{F}(X) \simeq C_*(X; A)$.

Notice that singular homology does of course make sense for manifolds. The properties that distinguish manifolds from general spaces, however, are not clearly evident in the definition of singular homology. One crucial point is that an n-manifold is built out of $\mathbb{R}^n$, whereas spaces in general are built out of points with no finer structure. Hence in a homology theory for manifolds we expect the Euclidean spaces to play a prominent role. Moreover we expect the excision property to be different, as we tend to cut manifolds along "collars." We offer the following temporarily vague definition of a homology theory for n-manifolds.

Definition 4. Let Mfld_n denote the category of n-manifolds (with some reasonable finiteness conditions) with embeddings. The category of homology theories for n-manifolds valued in a symmetric monoidal category $\mathcal{V}$ is the full subcategory

$$H(\mathsf{Mfld}_n, \mathcal{V}) \subset \mathsf{Fun}^{\otimes}(\mathsf{Mfld}_n, \mathcal{V})$$

of symmetric monoidal functors satisfying $\otimes$-excision along collar-gluings, i.e. if $M \cong M' \cup_{M_0 \times \mathbb{R}} M''$ then the canonical morphism

$$\mathcal{F}(M') \otimes_{\mathcal{F}(M_0 \times \mathbb{R})} \mathcal{F}(M'') \to \mathcal{F}(M)$$

is an equivalence in $\mathcal{V}$.

The central result of Ayala-Francis is the existence and uniqueness of a homology theory for manifolds (à la Eilenberg-Steenrod) known as factorization homology.

Theorem 5 (Ayala-Francis)**.** *There is an equivalence between homology theories for manifolds in $\mathcal{V}$ and n-disk algebras in $\mathcal{V}$,*

$$\mathrm{ev}_{\mathbb{R}^n} : H(\mathsf{Mfld}_n, \mathcal{V}) \rightleftarrows \mathsf{Alg}_{\mathsf{Disk}_n}(\mathcal{V}) : \int$$

where the equivalence is implemented by evaluation at $\mathbb{R}^n$ and factorization homology.

Ayala-Francis point out that a significant difference between the story for manifolds and spaces is that now the characterization in terms of $\mathcal{V}$ alone is replaced by the of n-disk algebras in $\mathcal{V}$. In other words, factorization homology is a machine that takes in an n-disk algebra and spits out a homology theory for manifolds.

We will not say anything here about the proof of this result but instead we will provide definitions and examples of n-disk algebras and computations of factorization homology in certain simple cases. One of the main corollaries of the above result is a vast generalization of Poincaré duality, which we will discuss below.

We begin with the definition of an n-disk algebra. Actually the theorem above already hints at what the definition should be: the restriction of any symmetric monoidal functor to the subcategory of Mfld_n consisting of disjoint unions of $\mathbb{R}^n$ completely determines the homology theory.

Definition 6. Define Disk_n to be the full subcategory of Mfld_n where the objects are finite disjoint unions of standard Euclidean spaces $\coprod_I \mathbb{R}^n$. An n-disk algebra in a symmetric monoidal category $\mathcal{V}$ is a symmetric monoidal functor $\mathsf{Disk}_n \to \mathcal{V}$.

Let's look at a few examples of n-disk algebras. The first example we use to give intuition on the algebraic structure of an n-disk algebra.

Example 7 (Commutative algebras)**.** Recall that a commutative dg algebra is a symmetric monoidal functor $A : \mathsf{Fin} \to \mathsf{Ch}$ from the category of finite sets under disjoint union to the category of chain complexes under tensor product. There is a natural symmetric monoidal functor

$$\begin{aligned} \pi_0 : \mathsf{Disk}_n &\to \mathsf{Fin} \\ \sqcup_I \mathbb{R}^n &\mapsto \pi_0(\sqcup_I \mathbb{R}^n) = I. \end{aligned}$$

Precomposing with π_0 now yields an n-disk algebra given any commutative dg algebra *for all* n. In other words every commutative dg algebra is an n-disk algebra for any n.

The idea here is that in an n-disk algebra there is not just one way of multiplying things — instead there are $\mathrm{Emb}(\mathbb{R}^n \sqcup \mathbb{R}^n, \mathbb{R}^n)$ ways of multiplying. In this example we see that applying π_0 remembers these various multiplications only up to homotopy leaving us with only the unique map from the two-point set to the one-point set.

The next example is the main object of study in nonabelian Poincaré duality.

Example 8 (n-fold loop spaces)**.** Let $(Z, *)$ be a pointed space. The n-fold loop spaces of Z yield n-disk algebras $\Omega^n Z$ in Spaces (or by postcomposing with $C_*(-;\mathbb{Z})$, in Ch) as follows. Recall that for M a space and Z a pointed space we say that a map $M \to Z$ is compactly supported if thee exists $K \subset M$ with K compact and such that $g|_{M \setminus K} = * \in Z$. Then we define

$$\Omega^n Z := \underset{c}{\mathrm{Maps}}(-, Z) : \mathsf{Disk}_n \to \mathsf{Spaces}.$$

Notice that compactly supported maps does indeed yield a contravariant functor (and that disjoint unions are taken to products). What does this have to do with the n-fold loop space? Notice that

$$\Omega^n Z = \mathrm{Maps}\left((D^n, \partial D^n), (Z, *)\right) \simeq \underset{c}{\mathrm{Maps}}(\mathbb{R}^n, Z),$$

where we identify $\mathbb{R}^n$ with the interior of the closed disk D^n.

For two more interesting examples (free n-disk algebras and n-disk enveloping algebras of Lie algebras) we refer to the online notes.

As we have all heard about $\mathcal{E}_n$-algebras, certain classes of *less* commutative algebras, and seen the examples above in that context, let us say a few words as to the relation between n-disk algebras and $\mathcal{E}_n$-algebras.

Definition 9. We define the category $\mathsf{Disk}_n^{\mathrm{rect}}$ to be the category consisting of finite disjoint unions of open unit disks $\sqcup_i D^n$ with morphisms rectilinear embeddings. An $\mathcal{E}_n$-algebra in $\mathcal{V}$ is a symmetric monoidal functor $\mathsf{Disk}_n^{\mathrm{rect}} \to \mathcal{V}$.

Recall that a rectilinear embedding is an embedding which can be written as a composition of translations and dilations. The main distinction between n-disk and $\mathcal{E}_n$-algebras is that n-disk algebras carry extra data about the automorphisms of $\mathbb{R}^n$. Rigidifying this data away by using framed embeddings (suitably homotopically defined) yields an equivalence of structures.

Lemma 10. *There is an equivalence* $\mathsf{Disk}_n^{fr} \to \mathsf{Disk}_n^{rect}$.

We refer to the notes for more details.

Exercise 11. Check that $\mathcal{E}_1$-algebras are, in a suitable sense, equivalent to associative algebras.

With these basic examples of gadgets that we will take as our coefficients in hand, we turn to the definition of factorization homology. We first need the category of n-disks in M.

Definition 12. Let M be an n-manifold. Define the category $\mathsf{Disk}_{n/M}$ to consist of n-disks together with embeddings into M, with morphisms (homotopy) commutative triangles.

We can now define factorization homology as a homotopy colimit (or left Kan extension).

Definition 13. Let M be an n-manifold and $A : \mathsf{Disk}_n \to \mathcal{V}$ be an n-disk algebra. Then the *factorization homology* of M with coefficients in A is

$$\int_M A := \mathrm{hocolim}\left(\mathsf{Disk}_{n/M} \to \mathsf{Disk}_n \xrightarrow{A} \mathcal{V}\right) \in \mathcal{V}.$$

Remark 14. Intuitively, factorization homology is a local-to-global machine. An n-manifold is built out of $\mathbb{R}^n$'s and an n-disk algebra assigns an object of $\mathcal{V}$ to each of these $\mathbb{R}^n$'s. Taking the homotopy colimit (or left Kan extension) is simply gluing these objects together in $\mathcal{V}$ using the blueprint that describes how M is glued together from $\mathbb{R}^n$'s.

To make this intuition concrete consider the very simple n-disk algebra valued in Spaces:

$$\begin{aligned} \mathrm{id} : \mathsf{Disk}_n &\to \mathsf{Spaces} \\ \sqcup_I \mathbb{R}^n &\mapsto \sqcup_I \mathbb{R}^n. \end{aligned}$$

We present our first computation of factorization homology.

Theorem 15. *Consider the identity functor* $\mathrm{id} : \mathsf{Disk}_n \to \mathsf{Spaces}$. *Then*

$$\int_M \mathrm{id} \simeq M.$$

The proof is rather nontrivial so we will say nothing about it here.

The next computation is a formal consequence of the previous theorem.

Corollary 16. *Consider the n-disk algebra valued in* Ch*,*

$$C_*(-;\mathbb{Z}) : \mathsf{Disk}_n \to \mathsf{Ch}$$
$$\sqcup_I \mathbb{R}^n \mapsto C_*(\sqcup_I \mathbb{R}^n;\mathbb{Z}) \simeq \mathbb{Z}^{\oplus I}.$$

This is often called a trivial n-disk algebra. Then

$$\int_M C_*(-;\mathbb{Z}) \simeq C_*(M;\mathbb{Z}),$$

i.e. the factorization homology of M with coefficients in the trivial n-disk algebra $\mathbb{Z}$ computes the singular homology of M.

Proof. Recall that the functor $C_*(-;\mathbb{Z}) : \mathsf{Spaces} \to \mathsf{Ch}$ has a right adjoint G such that $\pi_* GV$ is $H_* V$ if $* \geq 0$ and 0 otherwise. This is of course coming from the adjunction between simplicial sets and simplicial abelian groups, given by the free-forget functors. Hence $C_*(-;\mathbb{Z})$ preserves homotopy colimits. Since homotopy colimits commute with homotopy colimits,

$$\int_M C_*(-;\mathbb{Z}) \simeq C_*\left(\int_M \mathrm{id};\mathbb{Z}\right) \simeq C_*(M;\mathbb{Z})$$

by the theorem above. □

For another simple example that lines up with our intuition notice that $\mathsf{Disk}_{n/\mathbb{R}^n}$ has a final object. We thus obtain the following.

Proposition 17. *Let $A : \mathsf{Disk}_n \to \mathcal{V}$ be an n-disk algebra. Then the factorization homology of $\mathbb{R}^n$ with coefficients in A is just A again*

$$\int_{\mathbb{R}^n} A \simeq A.$$

As trivial as this computation may seem, we can now reconstruct a famous homology theory for associative algebras as factorization homology using a two-line argument.

Proposition 18. *Let $A : \mathsf{Disk}_1 \to \mathcal{V}$ be an associative algebra in a symmetric monoidal category. Then the factorization homology of the circle with coefficients in A is*

$$\int_{S^1} A \simeq HC_*(A),$$

the Hochschild homology of A.

Proof. We use the fact that factorization homology is $\otimes$-excisive. Write

$$S^1 \cong \mathbb{R} \cup_{\mathbb{R} \sqcup \mathbb{R}} \mathbb{R}$$

decomposing the circle as hemispheres. Then

$$\int_{S^1} A \simeq \int_{\mathbb{R}} A \otimes_{\int_{S^0 \times \mathbb{R}} A} \int_{\mathbb{R}} A \simeq A \otimes_{A \otimes A^{\mathrm{op}}} A \simeq HC_*(A).$$

Here we are using the previous proposition (and being closed-mouthed about the role of orientations). □

Remark 19. Notice that in the factorization homological construction of Hochschild homology the role of the circle is crystal clear. Not so for most definitions!

Another broad class of n-disk algebras was those that were actually commutative algebras. It turns out that commutative algebras, which are n-disk algebras for all n, are rather poor coefficients for homology theories for manifolds. In a sense they are too simple to remember that our manifold was more than just a topological space. More precisely, we have the following.

Theorem 20. *Let $A : \mathsf{Disk}_n \to \mathcal{V}$ be a commutative algebra. Then*

$$\int_M A \simeq \mathsf{Copow}(M, A),$$

i.e. the factorization homology of M with coefficients in A is the copowering (also known as tensoring) of the commutative algebra A with the underlying space of M.

Hopefully the above results emphasize the local-to-global nature of factorization homology. Its power, however, becomes especially apparent if we take as coefficients an n-fold loop space. The computation of the resulting factorization homology is known as nonabelian Poincaré duality.

Theorem 21 (Salvatore, Segal, Lurie, Ayala-Francis)**.** *Let M be an n-manifold and Z be a pointed $(n-1)$-connected space. Then the canonical map*

$$\int_M \Omega^n Z \xrightarrow{\sim} \underset{c}{\mathrm{Maps}}(M, Z)$$

is an equivalence of spaces.

In other words, the factorization homology of a compact manifold M with coefficients in the n-fold loop space of a pointed $(n-1)$-connected space Z computes the mapping space $\mathrm{Maps}(M, Z)$.

Remark 22. This result specializes to Poincaré duality between twisted homology and compactly support cohomology. Suppose $Z = K(A, i)$ is an Eilenberg-MacLane space for $i \geq n$. The theorem yields an equivalence of spaces

$$\int_M \underset{c}{\mathrm{Maps}}(-, K(A, i)) \simeq \underset{c}{\mathrm{Maps}}(M, K(A, i)).$$

Taking homotopy groups on the right,

$$\pi_j \underset{c}{\mathrm{Maps}}(M, K(A, i)) \cong [M, \Omega^j K(A, i)]_c \cong H^{i-j}(M; A).$$

To unpack the left, recall that the Dold-Kan functor (followed by geometric realization) takes $A[i] \mapsto K(A, i)$. Recall that the factorization homology of M with coefficients in the trivial n-disk algebra on the chain complex V computes $C_*(M; V)$. Now since $\Omega^n K(A, i) \simeq K(A, i-n)$ we find that

$$\int_M \Omega^n K(A, i) \simeq \left| \int_M A[i-n] \right| \simeq |C_*(M, A[i-n])|.$$

Hence on the left we have (shifted) homology.

One might object that the twist by orientation has not made an appearance here. The point is that we are working with n-disk algebras, not $\mathcal{E}_n$-algebras. In particular there is a nontrivial action of $O(n)$ on $K(A, i)$ and the $\mathbb{Z}/2$ coming from the components of $O(n)$ is giving us the orientation twist. In particular the left actually gives us twisted homology.

The proof of nonabelian Poincaré duality is relatively straightforward given the characterization of factorization homology as the unique homology theory for manifolds.

Proof of nonabelian Poincaré duality. By Ayala-Francis lemma 4.5, $\mathrm{Maps}_c(-, Z)$ is a homology theory for manifolds valued in Spaces. The connectivity conditions on Z are used in the proof of this lemma. The Ayala-Francis characterization for homology theories for manifolds now implies that $\mathrm{Maps}_c(-, Z)$ is equivalent to factorization homology with coefficients in $\mathrm{Maps}_c(\mathbb{R}^n, Z)$. But $\mathrm{Maps}_c(\mathbb{R}^n, Z) \simeq \Omega^n Z$, whence

$$\int_M \Omega^n Z \simeq \underset{c}{\mathrm{Maps}}(-, Z).$$

□