COMPLEX GEOMETRY

MIHNEA POPA

Contents

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Plan for the course:

(1) brief review of holomorphic functions and functions of several complex variables
(2) complex manifolds, analytic sets, (p, q)-forms, sheaves, cohomology
(3) (compact) Kähler manifolds, e.g. projective manifolds
(4) main theorems: Hodge theorem, Kodaira embedding, Chow's theorem

We'll start by reviewing the local theory, for which Gunning-Rossi is a good reference.

Definition 1. Suppose $U \subset \mathbb{C}^n$ is an open set. We say that $f : U \to \mathbb{C}$ is **holomorphic on** U if for each $x \in U$ there exists an open $V \subset U$ such that $x \in V$

and on V we can write f as a convergent power series

$$f(z) = f(z_1, \cdots, z_n) = \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n}(z_1 - x_1)^{i_1} \cdots (z_n - x_n)^{i_n}.$$

We denote the $\mathbb{C}$-algebra of holomorphic functions on U as $\mathcal{O}(U)$.

Given a map $f : U \to W$ where $U \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ are open, we say that f is holomorphic if the compositions with the coordinate functions $f_1, \ldots, f_m$ are holomorphic.

How do we determine when a function of several variables is holomorphic?

Lemma 2 (Osgood's lemma)**.** *We have that f is holomorphic if and only if f is continuous and is holomorphic in each variable separately, i.e. if $(p_1, \ldots, z, \ldots, p_n)$ (with z in the jth position varying) are points in U, we require that $f(p_1, \ldots, z, \ldots, p_n)$ is holomorphic in z.*

Proof. Left as an easy exercise. □

Remark 3. The above lemma is true even without the continuity condition, but is more difficult. This is a theorem of Hartog's.

Recall that in one-variable complex analysis we detect holomorphicity via the Cauchy-Riemann equations. In particular, given $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$ continuously differentiable we can write $f = f_1 + if_2$, and

$$f \text{ is holomorphic} \iff \frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \text{ and } \frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}.$$

If you can never remember these formulas, it is helpful to use instead:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

under which

$$f \text{ is holomorphic} \iff \frac{\partial f}{\partial \bar{z}} = 0.$$

This generalizes easily to multiple variables by the above lemma where we as that $\partial f / \partial \bar{z}_i = 0$ for all i.

Exercise 4. Use this to observe that in the expansion $f(z) = \sum_I a_I \cdot (z - x)^I$ the coefficients can be computed

$$a_{i_1 \ldots i_n} = \frac{1}{i_1! \cdots i_n!} \frac{\partial^I f}{\partial z_1^{i^1} \cdots \partial z_n^{i_n}}(x).$$

This is an example of a result that is similar to the one-variable case. There are however results that are quite different in the higher dimensional cases. Let's first go over some of the similarities.

Theorem 5 (Identity principle)**.** *Let $U \subset \mathbb{C}^n$ be a connected open set and let $f, g : U \to \mathbb{C}$ be holomorphic. If $f(z) = g(z)$ for every z in some open $V \subset U$ with $z \in V$ then $f(z) = g(z)$ for all $z \in U$.*

Theorem 6 (Maximum principle)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function. If $z_0 \in U$ is such that $|f(z_0)| \geq f(z)|$ for all $z \in U$ then f is constant.*

Here, on the other hand, is something completely new that only occurs in high dimensions.

Theorem 7 (Hartog's extension theorem)**.** *Let $U \subset \mathbb{C}^n$ be an open set for $n \geq 2$. Then every holomorphic function $f : U \setminus Z \to \mathbb{C}$, where Z is a closed subset of (complex) codimension at least 2, extends to a holomorphic function everywhere.*

Proof. See, for instance Gunning-Rossi. □

In particular every holomorphic function defined away from a single point extends to a holomorphic function on U. This of course not true for $n = 1$: there exist functions with essential singularities. This theorem is also true in algebraic geometry—in the case where we have a regular function on a smooth algebraic variety.

Let's now turn to the very local picture: germs of holomorphic functions and localizations of rings.

Definition 8. Let $z_0 \in U, V \subset \mathbb{C}^n$ with $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. We say that f and g are equivalent if there exists an open set W such that $f|_W = g|_W$. The equivalence class of f is called the **germ of f at z** (this is an equivalence relation on the set of pairs of opens containing z and functions on the open). The notation we will use is $\mathcal{O}_{z_0} = \mathcal{O}_{\mathbb{C}^n, z_0} = \mathcal{O}_{U, z_0}$.

Remark 9. More formally, in terms of sheaf theory, there exists a restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$, so the opens containing a point z form a directed set, and

$$\mathcal{O}_{z_0} = \varinjlim_{z_0 \in U} \mathcal{O}(U).$$

In other words the set of germs is just the stalk of the sheaf of holomorphic function. Notice of course that the only value of a germ that makes sense is at z itself.

Remark 10. We have an isomorphism

$$\mathcal{O}_{z_0} \cong \mathbb{C}\{z_1, \ldots, z_n\},$$

the ring of convergent power series. To see this, we may as well assume that $z = 0$. Then in a neighborhood of 0, by definition of holomorphicity, f can be written as a convergent power series.

How nice are these rings of germs? If $n = 1$, every function can be written as

$$f(z) = z^k g(z)$$

where $g(0) \neq 0$. Notice that g is a unit (as a germ). Hence $\mathbb{C}\{z\}$ is a DVR with maximal ideal generated by z. What about higher dimensions? Let $n \geq 2$. Let's just call $\mathcal{O}_n = \mathcal{O}_{z_0}$ since all points look the same. The first thing to notice is that $\mathcal{O}_n$ is a local ring, i.e. it has only one maximal ideal. This ideal is $\mathfrak{m}_n = \{f \in \mathcal{O}_n \mid f(0) = 0\}$, since everything outside of this ideal is a unit. Of course, $\mathfrak{m}_n = (z_1, \ldots, z_n)$. In particular

$$f \notin \mathfrak{m}_n \iff f(0) \neq 0 \iff 1/f \text{ holc near } 0.$$

Lemma 11. *We have that*

$$\dim_{\mathbb{C}} \mathfrak{m}_n / \mathfrak{m}_n^2 = n.$$

Proof. Notice that

$$\mathfrak{m}_n/\mathfrak{m}_n^2 \cong \mathfrak{m}_n \otimes_{\mathcal{O}_n} \mathcal{O}_n/\mathfrak{m}_n \cong \mathbb{C}^n$$

since $\mathfrak{m}_n$ has n generators. □

Lemma 12. *$\mathcal{O}_n$ is a domain.*

Proof. Obvious from the description as convergent power series. Also follows from the identity principle. □

We will show that $\mathcal{O}_n$ is a Noetherian UFD. To do this, we need the Weierstrass preparation theorem. We're running out of time for today but let's at least introduce some notation. Consider the chain of inclusions

$$\mathcal{O}_{n-1} \subset \mathcal{O}_{n-1}[z_n] \subset \mathcal{O}_n$$

where the set in the middle consists of monic polynomials. WPT will tell us when functions in the largest set are actually in the middle set. This will yield the Noetherian property. WDT, on the other hand, will yield the UFD property. More explicitly $h \in \mathcal{O}_{n-1}[z_n]$ is written

$$h = z_n^d + c_{d-1}z_n^{d-1} + \cdots + c_1 z_n + c_0$$

where $c_i \in \mathcal{O}_{n-1}$.

Definition 13. We say that h is a **Weierstrass polynomial** if $c_0, \dots, c_{d-1} \in \mathfrak{m}_{n-1}$, i.e. $c_i(0) = 0$.

Definition 14. Given $0 \in U \subset \mathbb{C}^n$ with U open and $f \in \mathcal{O}(U)$, we say that f is **regular in** z_n if $f(0, z_n) \not\equiv 0$. In this case $f(0, z_n) = z_n^k \cdot u(z_n)$ for u invertible, and we say that f is regular of order k.

Example 15. If h is a Weierstrass polynomial of degree d then h is regular of degree d. Indeed, $h(0, z_n) = z_n^d$.

The WPT will tell us (up to units) that every regular function will be a Weierstrass polynomial.

2. January 19, 2017

Recall last time we showed that $\mathcal{O}_n$, the ring of germs of holomorphic functions (convergent power series), is a local domain. To obtain more results we were going to use some results from several complex variables. Recall the notions of Weierstrass polynomial and regular germ from last time. Weierstrass polynomials of degree d are obvious examples of regular functions of order d.

Theorem 16 (Weierstrass preparation)**.** *Let $f \in$ mathcalO$_n$ be regular of order d in z_n. Then there exists a unique Weierstrass polynomial $h \in \mathcal{O}_{n-1}[z_n]$ of degree d such that $f = u \cdot h$ for u a unit in $\mathcal{O}_n$.*

This might seem weak at first sight. For instance one might take $f(z) = z_i$. This is certainly not regular in z_n. It turns out, however, that by changing coordinates one can always make a (collection of) function(s) regular in a fixed variable.

Lemma 17. *Given $f_1, \ldots, f_k \in \mathcal{O}_n \setminus \{0\}$, there exists a linear change of coordinates which makes them all regular in z_n.*

Proof. We may as well assume $k = 1$: take $f = f_1 \cdots f_k \neq 0$. Now there exists some $w_0 \in \mathbb{C}^n$ such that $f(w_0) \neq 0$. Hence $f(tw_0) \neq 0$ for t in some open set $V \subset \mathbb{C}$. Change coordinates,say linearly, such that $w_0 = (0, \ldots, 0, 1)$. Hence

$$f(0, z_n) = f(z_n \cdot w_0) \not\equiv 0.$$

□

Corollary 18. *The local ring $\mathcal{O}_n$ is a UFD.*

Proof. We induct on n. The case $n = 0$ is clear since $\mathcal{O}_0 \cong \mathbb{C}$. Assume that $\mathcal{O}_n$ is a UFD. Then by the Gauss lemma we find that $\mathcal{O}_{n-1}[z_n]$ is also a UFD. If we now take $f \in \mathcal{O}_n$, by the WPT we have that $f = u \cdot h$ where h is a Weierstrass polynomial, after change of coordinates. Say $f = f_1 \cdot f_2 \in \mathcal{O}_n$ where each of these have decompositions (in the same coordinates by the above lemma) $f_1 = u_1 h_1$ and $f_2 = u_2 h_2$. Now

$$uh = f = f_1 f_2 = u_1 u_2 \cdot h_1 h_2$$

whence we find that $h = h_1 h_2$. In other words, it suffices to show uniqueness of factorization in the ring of Weierstrass polynomials, which, as above follows inductively via the Gauss lemma. □

We still haven't shown the most basic property, that $\mathcal{O}_n$ is Noetherian. For this we need the following.

Theorem 19 (Weierstrass division)**.** *Let $h \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree d. Then for $f \in \mathcal{O}_n$ there exists a unique formula*

$$f = q \cdot h + r$$

with $q \in \mathcal{O}_n$ and $r \in \mathcal{O}_{n-1}[z_n]$ of degree strictly less than d. Moreover, if $f \in \mathcal{O}_{n-1}[z_n]$ then $q \in \mathcal{O}_{n-1}[z_n]$ as well.

Corollary 20. *The local UFD $\mathcal{O}_n$ is a Noetherian ring.*

Proof. Again we will induct on n. For $n = 0$ the result is clear. Suppose now that $\mathcal{O}_{n-1}$ is Noetherian. Let $I \subset \mathcal{O}_n$ be an ideal. We will show that this ideal is generated by a finite number of elements. Pick any $h \in I$ and change coordinates such that h is regular in z_n of order d. We can assume that, in these coordinates, up to a unit, h is a Weierstrass polynomial. For any $f \in I$ we have $f = qh + r$ by the theorem above. It follows that $r \in I$ whence $I = (h) + J$ for some other ideal J. In particular $J = I \cap \mathcal{O}_{n-1}[z_n]$. But now J is finitely generated by induction (by the Hilbert basis theorem $\mathcal{O}_{n-1}[z_n]$ is Noetherian because $\mathcal{O}_{n-1}$ is) and (h) is of course generated by a single element. □

In summary, germs of holomorphic functions behave very much like polynomials.

2.1. **Analytic sets.** Let us now discuss the analog of affine algebraic sets.

Definition 21. Let $U \subset \mathbb{C}^n$ be an open set. A subset $Z \subset U$ is called **analytic** if for every $z \in Z$ there exists an open set $V \subset U$ containing z such that $V \cap Z$ is the zero locus of a collection of holomorphic functions.

Notice that there is a crucial difference with the Zariski topology in the sense that an analytic set is not *globally* cut out by holomorphic functions.

Remark 22.
(1) Clearly an analytic set Z is closed.
(2) If $n = 1$ then Z is a set of isolated points.
(3) For $f \in \mathcal{O}(U)$ we call $Z = Z(f)$ an **analytic hypersurface**.

Fix $z_0 \in Z$ a point on an analytic set. By coordinate change we may assume that $z_0 = 0$.

Definition 23. We define the **ideal of** Z in $\mathcal{O}_n$ to be

$$I(Z) = \{f \in \mathcal{O}_n \mid f|_Z \equiv 0\} \subset \mathcal{O}_n.$$

One should be a little careful here and think of germs of sets in this definition.

Remark 24.
(1) If $Z_1 \subset Z_2$ then $I(Z_2) \subset I(Z_1)$.
(2) We have that $Z(I(Z)) = Z$, where by the first Z we mean the common zero locus of elements in $I(Z)$.
(3) We have that $I(Z) = (f_1, \ldots, f_r)$ since $\mathcal{O}_n$ is Noetherian. In other words, every analytic set is locally given by finitely many functions.

Definition 25. An analytic set is **(locally) reducible** if it can be (locally) written as the union of two nontrivial analytic sets $Z = Z_1 \cup Z_2$.

Example 26. Consider the equation $x^2 = y^2$. This clearly gives us something reducible. We call this a node.

But you can have another type of node like $y^2 = x^3 + x^2$. In the analytic topology, we have a small neighborhood of the self-intersection point where the node appears reducible.

Proposition 27. *Let $0 \in Z$ be an analytic set. Then, in some neighborhood of zero, there exists a decomposition $Z = Z_1 \cup \cdots \cup Z_r$ with Z_i irreducible analytic sets. If we assume that $Z_i \not\subset Z_j$ for all i, j then this decomposition is unique (up to reordering).*

Proof. Suppose not. Then Z decomposes $Z = Z_1 \cup Z_1'$ but again now say Z_1 is irreducible whence Z_1' is not, and so on indefinitely. This gives us an infinite chain of proper inclusions

$$Z \supsetneq Z_1 \supsetneq Z_2 \supsetneq \cdots$$

This corresponds to

$$I(Z) \subsetneq I(Z_1) \subsetneq I(Z_2) \subsetneq \cdots$$

which contradicts the Noetherian condition of $\mathcal{O}_n$. For the uniqueness say $Z = Z_1' \cup \cdots \cup Z_k'$. For every i we have $Z_i' = (Z_i' \cap Z_1) \cup \cdots \cup (Z_i' \cap Z_r)$. There are all analytic sets but the left is irreducible whence it must be contained in another Z_j. Performing the argument the other way around we find that $Z_j \subset Z_\ell'$ for some ℓ for some ℓ. This forces us to have $Z_i' = Z_\ell' = Z_j$. By renumbering and throwing away, we can continue by induction on r. □

Next time we will state the Nullstellensatz and start discussing complex manifolds.

3. January 22, 2018

We will have a makeup lecture tomorrow in Lunt 105 from 2-4.

Finishing up from last time, let us state the Nullstellensatz which is true in the setting of analytic sets as well.

Theorem 28 (Nullstellensatz)**.** *We have that* $I(Z(I)) = \operatorname{rad} I$*, where*

$$\operatorname{rad} I = \{f \in \mathcal{O}_n \mid \exists k \geq 1, f^k \in I\}.$$

3.1. Complex manifolds.

We now turn to the notion of a complex manifold. We will assume familiarity with the theory of smooth manifolds. Let X be a topological space, Hausdorff with a countable basis of open subsets. For each $U \subset X$ we will denote by $C(U)$ the ring of continuous functions $f : U \to \mathbb{C}$.

Definition 29. A **geometric structure** $\mathcal{O}_X$ on X is an assignment of subrings $\mathcal{O}_X(U) \subset C(U)$ for every $U \subset X$ such that

(1) the constant functions are include: $\mathbb{C} \subset \mathcal{O}_X(U)$,
(2) if $f \in \mathcal{O}_X(U)$ and $V \subset U$ is open then $f|_V \in \mathcal{O}_X(V)$,
(3) if $f_i \in \mathcal{O}_X(U_i)$ (for $i \in I$ some index set) such that

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for each $i, j \in I$, then denoting $U = \cup_{i \in I} U_i$, there exists a unique $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$.

Remark 30. This geometric structure, in other words, is a choice of subsheaf of the sheaf of continuous functions $\mathcal{O}_X \subset C_X$. We will call the pair $(X, \mathcal{O}_X)$ a **geometric space** or a **ringed space**.

Example 31. The main examples for us will be the sheaves of smooth and holomorphic functions (and eventually algebraic functions).

More explicitly, for $X \subset \mathbb{R}^n$ open, then to $U \subset X$ we assign $\mathcal{A}_X(U) \subset C(U)$ the ring of smooth functions on U.

For $X \subset \mathbb{C}^n$ open, then to $U \subset X$ we assign $\mathcal{O}_X(U) \subset C(U)$ the ring of holomorphic functions on U.

Definition 32. A morphism of geometric spaces is $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for every open set $U \subset Y$, if $g \in \mathcal{O}_Y(U)$ then $g \circ f \in \mathcal{O}_X(f^{-1}U)$. This yields maps

$$f_U^* \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

of rings (compatible with restriction). Notice that f is an isomorphism if it has an inverse which is also a morphism.

Exercise 33. Given a map $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ then $f : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ is a morphism of geometric spaces if and only if f is holomorphic. Use the lemma that f is holomorphic if and only if $g \circ f$ is holomorphic for every $g : V \to \mathbb{C}$ holomorphic.

Given a geometric space $(X, \mathcal{O}_X)$ and a fixed $U \subset X$ open then we obtain a geometric space and a map $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ in the obvious way.

Now we come to the main definition.

Definition 34. A **complex manifold** is a geometric space $(X, \mathcal{O}_X)$ such that every $x \in X$ has a neighborhood $x \in U \subset X$ such that the induced geometric structure is isomorphic (as a geometric space) to $(V, \mathcal{O}_V)$ which is a geometric subspace of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$.

Remark 35. We can now define directly,

$$\mathcal{O}_{X,x} = \operatorname*{colim}_{U \ni x} \mathcal{O}_X(U) \supset \mathfrak{m}_x,$$

where $\mathfrak{m}_x$ are the functions vanishing at x. This is of course nothing new for manifolds due to their local characterization. As usual we may define the dimension

$$n = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2.$$

Certainly the dimension is locally constant, and constant if X is connected.

There is of course the more classical definition of complex manifolds via charts and atlases. Recall that an **atlas** on X is the data of an open cover $\{U_i\}_{i\in I}$ of X with homeomorphisms $\phi_i : U_i \to V_i$ for V_i open in $\mathbb{C}^n$, such that for each $i, j \in I$ the transition functions

$$g_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are holomorphic isomorphisms, i.e. biholomorphic. Then a manifold is defined as an equivalence class of atlases, which we won't outline in detail.

Exercise 36. Show that the two definitions coincide.

Example 37. We now turn to some basic examples.

(1) An open set $U \subset \mathbb{C}^n$ is a complex manifold.
(2) A Riemann surface is a connected one-dimensional complex manifold.
(3) The following, projective space, is a compact complex manifold $\mathbb{P}^n_{\mathbb{C}}$. As a set it is just the set of lines through the origin in $\mathbb{C}^{n+1}$. We can write this as $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/(\mathbb{C}^\times \setminus \{0\})$, i.e. $x \sim y$ if and only if there exists $\lambda \in \mathbb{C}^\times$ such that $x = \lambda y$. We thus have the projection map

$$\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{q} \mathbb{P}^n$$

which we use to endow $\mathbb{P}^n$ with the quotient topology (recall that this means $U \subset \mathbb{P}^n$ is open if and only if $q^{-1}(U)$ is open). We leave it as an exercise to check that $\mathbb{P}^n$ is Hausdorff and compact. The compactness follows from the fact that we can write it as a quotient of a compact space. We will denote points in projective space as

$$(x_0 : \cdots : x_n) \in \mathbb{P}^n$$

called homogeneous coordinates. We have that

$$(x_0 : \cdots : x_n) = (\lambda x_0 : \cdots : \lambda x_n).$$

for $\lambda \neq 0$.

We now give $\mathbb{P}^n$ the structure of a complex manifold. Simply define

$$\mathcal{O}_{\mathbb{P}^n}(U) = \{f \in C(U) \mid f \circ q \text{ is holomorphic on } q^{-1}(U)\}.$$

We now use a standard atlas for $\mathbb{P}^n$. For $i = 0, \ldots, n$ define

$$U_i = \{x \in \mathbb{P}^n \mid x_i \neq 0\}$$

and notice that we have maps

$$\phi_i : U_i \to \mathbb{C}^n$$
$$(x_0 : \cdots : x_i \neq 0 : \cdots x_n) \mapsto (\frac{x_0}{x_i}, \ldots, \hat{x}_i, \ldots, \frac{x_n}{x_i}).$$

These are homeomorphisms as we have invese

$$\phi_i^{-1}(z_1, \ldots, z_n) \mapsto (z_1 : \cdots 1 : \cdots z_n).$$

The maps ϕ_i establish an isomorphism between $(U_i, \mathcal{O}_{\mathbb{P}^n}|_{U_i}) \cong (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. In other words, ϕ_i induces isomorphisms

$$\mathcal{O}_{\mathbb{C}^n}(U) \to \mathcal{O}_{\mathbb{P}^n}(\phi_i^{-1}U).$$

This is because $f : U \to \mathbb{C}$ is holomorphic if and only if $f \circ \phi_i q$ is holomorphic. This is clear from the holomorphic expressions we have written for these maps. Notice that projective space is a special case of the next example.

(4) The next class of complex manifolds we consider is that of group quotients. Not every group action will yield a complex manifold, so we need to do a bit of work. Notice first that the automorphisms of X (biholomorphisms $X \to X$) form a group. Our groups G that we will quotient by will be subgroups $G \subset \operatorname{Aut} X$.

4. January 23, 2018

4.1. **More examples of complex manifolds.** By the way, the first homework is now online. Turn it in in about two weeks, say.

Consider a subgroup $G \subset \operatorname{Aut} X$ for X a complex manifold.

Definition 38. We say that G acts **properly discontinuously** on X if for each pair of compact subsets $K_1, K_2 \subset X$ there exist only finitely many $\phi \in G$ such that $\phi(K_1) \cap K_2 \neq \varnothing$.

We say that G acts **freely** (or without fixed points) if $\phi(x) = x$ for some x then $\phi = \mathrm{id}$.

Example 39. Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e. a discrete subgroup isomorphic to $\mathbb{Z}^2 \subset \mathbb{C} \cong \mathbb{R}^2$. After a coordinate change we may assume that the generators of this lattice are $1, \tau \in \mathbb{C}$. There is a standard action of this lattice on $\mathbb{C}$ by translation:

$$a \in \Lambda \mapsto \phi_a(x) = x + a \qquad x \in \mathbb{C}.$$

It is easy to check that this action is properly discontinuous and free.

Consider the quotient

$$X/G = \{[x] \mid y \in [x] \iff \exists \phi \in G, \phi(x) = y\}$$

as a set. There is an evident map $q : X \to X/G$ whence we equip X/G with the quotient topology.

Lemma 40. *The map $q : X \to X/G$ is an open mapping.*

Proof. Given $U \subset X$ open we'd like to show that $q(U)$ is open. By definition of the quotient topology, $q(U)$ is open if and only if $q^{-1}(q(U))$ is open. But this last set is just

$$q^{-1}(q(U)) = \cup_{\phi \in G} \phi(U).$$

But $\phi(U)$ is open since ϕ is a diffeomorphism, and we are done. □

The following result shows that it is surprisingly easy to check when the quotient is again a complex manifold, unlike in the algebraic case where a whole geometric invariant theory has to be developed.

Proposition 41. *If G acts properly discontinuously and freely then X/G is a complex manifold and the map $q : X \to X/G$ is holomorphic and locally biholomorphic.*

Proof. See homework exercises. □

It turns out that

$$\mathcal{O}_{X/G}(U) = \mathcal{O}_X(q^{-1}U)^G := \{f \in \mathcal{O}_X(q^{-1}(U)) \mid f \circ \phi = f, \forall \phi \in G\},$$

the invariants, as one expects.

Example 42 (Complex tori)**.** In the example above of the lattice and the translation action we obtain $E = \mathbb{C}/\Lambda$, which we call an elliptic curve. More generally we can take lattices $\Lambda \subset \mathbb{C}^n$ acting by translation to form the quotient $\mathbb{C}^n/\Lambda$, which we call a complex torus. These complex manifolds are compact: choose $\lambda_1, \ldots, \lambda_{2n}$ generators for Λ. We have a surjection $[0,1]^n \to \mathbb{C}^n/\Lambda$ sending $(a_1, \ldots, a_n) \mapsto [\sum a_i \lambda_i]$. It turns out that these higher-dimensional need not necessarily be algebraic (unlike the case of elliptic curves). There will be conditions known as the Hodge-Riemann bilinear relations.

Example 43 (Hopf manifolds)**.** The following are examples of complex manifolds that will not be Kähler. Let $0 < \lambda < 1$ be a real number. We define an action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ given by

$$(k, (z_1, \ldots, z_n)) \mapsto (\lambda^k z_1, \ldots, \lambda^k z_n).$$

One checks that the action is free and properly discontinuous so we obtain a complex manifold. Writing $X = \mathbb{C}^n \setminus \{0\}/\mathbb{Z}$ for the quotient, it turns out that X is diffeomorphic to $S^{2n-1} \times S^1$. For $n = 1$ it turns out that we get an elliptic curve, but for $n \geq 2$ we will see later that there is no Kähler structure.

The next major example is that of affine and projective hypersurfaces.

Recall that we say a point y in the codomain of a holomorphic map f is a regular value if at every point x in the preimage, the differential $df(x)$ is surjective. In the case of a function (mapping into $\mathbb{C}$) we simply need that there exists an i such that $\partial_{z_i} f(x) \neq 0$ for all $x \in f^{-1}(y)$.

Lemma 44. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic with 0 as a regular value for f. Then $X = f^{-1}(0)$ is a complex (sub)manifold (of $\mathbb{C}^n$).*

Let $Y \subset X$ be a subset of a complex manifold. Y is certainyl a topological space under the induced topology, but moreover we can define for every open $V \subset Y$,

$$\begin{aligned} \mathcal{O}_X|_Y(V) = &\{f : V \to \mathbb{C} \mid \forall y \in Y, \exists y \in U_y \text{ open in } X, \exists f_y \in \mathcal{O}_X(U_y), \\ &f(v) = f_y(v) \forall v \in V \cap U_y\}. \end{aligned}$$

This yields a geometric structure on Y.

Example 45. The main example is of course that of linear subspaces $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$ where we send $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$. In this case there is a standard extension of any function $f : V \to \mathbb{C}$ for $V \subset \mathbb{C}^k$ to a function on $V \times \mathbb{C}^{n-k}$. In other words $\mathcal{O}_{\mathbb{C}^n}|_{\mathbb{C}^k} = \mathcal{O}_{\mathbb{C}^k}$.

Definition 46. A subset $Y \subset X$ for X a complex manifold is a **complex submanifold** of dimension k if for every $y \in Y$ there exists a chart of X around y, call it (U, ϕ) where $\phi : U \xrightarrow{\sim} V \subset \mathbb{C}^n$ for V open, such that $\phi(U \cap Y) = V \cap \mathbb{C}^k \subset \mathbb{C}^n$. We take $(Y, \mathcal{O}_X|_Y)$ to be the geometric structure on Y.

Assume now that $F \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogeneous polynomial of degree d. We get an induced map $F : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$. Suppose 0 is a regular value for F. Then $F^{-1}(0) \subset \mathbb{C}^{n+1} \setminus \{0\}$ is a complex submanifold. Moreover, by homogeneity, we have that

$$X = F^{-1}(0)/\mathbb{C}^\times \subset \mathbb{P}^n .$$

We leave it as an exercise to show that X is a submanifold of $\mathbb{P}^n$ of dimension $n-1$ (use the dehomogeneization of F).

Definition 47. If $f_1, \ldots, f_k : \mathbb{C}^n \to \mathbb{C}$ are holomorphic and 0 is a regular value for $(f_1, \ldots, f_k) \to \mathbb{C}^k$. In this case $X = Z(f_1) \cap \cdots \cap Z(f_k) \subset \mathbb{C}^n$ is a submanifold of dimension $n - k$, called the **complete intersection** of $f_1, \ldots, f_k$.

Similarly if the f_i are homogeneous, we obtain a complete intersection in $\mathbb{P}^{n-1}$.

Notice that the regular value property is the same as if we were to do this complete intersection sequentially. In algebra this is related to regular sequences, Cohen-Macaulay modules, and so on.

Another huge class of examples is furnished by complex Lie groups. These are complex manifolds G given a group structure via holomorphic maps $m : G \times G \to G$ and $i : G \to G$.

Example 48. The complex tori defined above $\mathbb{C}^n/\Lambda$ are compact complex Lie groups.

Here is an amusing exercise. Assume abstractly that G is a compact complex Lie group and show that it is commutative.

Example 49. The usual first example is that of $M_n(\mathbb{C})$ the $n \times n$-matrices under addition. Inside this group are Lie groups under multiplication. For instance we have $GL_n(\mathbb{C})$ consisting of all nonsingular matrices. That the inverse is a holomorphic map comes from Cramer's formula. Similarly we have $SL_n(\mathbb{C})$, the matrices of determinant 1 and the symplectic groups $Sp_{2n}(\mathbb{C}) = \{A \mid {}^tA\Omega A = \Omega\}$ where

$$\Omega = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & - \end{pmatrix}$$

which are subsets of $M_{2n}(\mathbb{C})$ of dimension $n(2n+1)$. One has to be a bit careful because there exist real Lie groups in $GL_n(\mathbb{C})$ that are not complex, such as U_n or SU_n.

These are all examples of nonabelian, noncompact, "linear groups."

4.2. **Blow-ups.** The next example is that of blow-ups. Let X be a complex manifold of dimension n and $x \in X$ be a fixed point. We wish to construct $\tilde{X} = \mathrm{Bl}_x X \xrightarrow{\pi} X$ that replaces x by a copy of $\mathbb{P}^{n-1}$ but leaves the rest of X unchanged. This is some sort of surgery operation in the category of complex manifolds or algebraic varieties which usually allows you to get rid of certain bad phenomenon such as singularities, and hence will be very useful later on.

The main case is as follows. We take $X = \mathbb{C}^n$ and $x = 0$ the origin.

Definition 50. We define the **blow-up** of $\mathbb{C}^n$ at the origin as

$$\mathrm{Bl}_0\, \mathbb{C}^n = \{(z, L) \mid z \in L\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$$

where we think of the points of the projective space as lines passing through the origin in our copy of $\mathbb{C}^n$. This is what is called an incidence correspondence.

There is a natural projection map $\pi : \mathrm{Bl}_0\, \mathbb{C}^n \to \mathbb{C}^n$ given by the projection to the first factor. Notice that the preimage of z under this projection is a single point if $z \neq 0$. If $z = 0$ then $\pi^{-1}(0)$ is the set of lines through the origin, i.e. $\mathbb{P}^{n-1}$. This preimage is known as the **exceptional locus**.

While this is a nice picture to have, it is useful to have some explicit equations.

Proposition 51. *The blow-up* $\mathrm{Bl}_0\, \mathbb{C}^n$ *is a complex manifold of dimension* n *and the map* π *is holomorphic. The exceptional locus* $\pi^{-1}(0)$ *is a submanifold of dimension* $n-1$.

Proof. Let's use coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ and homogeneous coordinates $[a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}$. The incidence condition is that a point z belong to a line a. This happens if and only if

$$z_i a_j = z_j a_i$$

for each i, j. We call these equations the defining equations of the blow-up.

Now write $q : \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{n-1}$ for the projection onto the second factor. Recall that we have explicit coordinate charts $U_i \subset \mathbb{P}^{n-1}$ given as the points $[a_1 : \cdots : a_n]$ such that $a_i \neq 0$. We have an open cover of the blow-up by $\{V_i\}$ where $V_i = q^{-1}(U_i)$. We claim that each of these V_i is biholomorphic to $\mathbb{C}^n$. Explicitly,

$$V_i = \{(z, a) \mid a_i \neq 0, z_j = z_i a_j / a_i \text{ for } j \neq i\}.$$

We define f_i to send

$$(z, a) \mapsto (\frac{a_1}{a_i}, \dots \frac{a_{i-1}}{a_i}, z_i, \frac{a_{i+1}}{z_i}, \dots, \frac{a_n}{a_i})$$

which has inverse

$$(y_1, \dots, y_n) \mapsto (z, a)$$

where

$$\begin{aligned} a &= (y_1 : \cdots : y_{i-1} : 1 : \cdots : y_n) \\ z &= y_i \cdot a. \end{aligned}$$

The transition functions are $g_{i,j} = f_i \circ f_j^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ given $g_{i,j}(y_1, \dots, y_n) = (w_1, \dots, w_n)$ where

$$w_k = \begin{cases} y_k / y_i, & k \neq i, j \\ y_i y_j, & k = i \\ 1/y_i, & k = j. \end{cases}$$

Notice that this makes sense because on the transition domain we have that y_i is nonzero. These expressions are holomorphic whence we obtain a complex manifold.

If we now apply π in our charts

$$(\pi \circ f_i^{-1})(y_1, \dots, y_n) = (y_i y_1, \dots, y_i y_{i-1}, y_i, y_i y_{i+1}, \dots, y_i y_n)$$

whence we see that π is holomorphic. Finally notice that $\pi^{-1}(0) \cap V_i$ corresponds under f_i to the linear hypersurface $y_i = 0$ in $\mathbb{C}^n$. Hence $\pi^{-1}(0)$ is a submanifold of dimension $n - 1$. Moreover we obtain standard coordinates on the exceptional locus. □

Remark 52. Notice that the blow-up construction above can be applied to an open set $U \subset \mathbb{C}^n$ as $\operatorname{Bl}_0 U = \pi^{-1} U \xrightarrow{\pi} U$. In particular it will turn out that this blow-up is the total space of a certain rank 1 vector bundle.

We now turn to the blow-up of a manifold in general. Let X be a complex manifold and $x \in X$ a point. Fix a coordinate chart $x \in U \xrightarrow{f} V \subset \mathbb{C}^n$ sending $x \mapsto 0$. Notice that $U \setminus \{x\} \cong V \setminus \{0\} \cong \tilde{V} - \pi^{-1}(0)$ since the blow-up is identical to the original space away from the chosen point. Now we will glue $X \setminus \{x\}$ and $\tilde{V}$ along $U \setminus \{x\}$. In particular we consider

$$\operatorname{Bl}_x X := (X \setminus \{x\}) \coprod \tilde{V} /$$

where the equivalence relation identifies

$$y \in U \setminus \{x\} \sim w \in \tilde{V} \iff f(y) = \pi(w).$$

By the holomorphicity of f and π it will follow that the transition functions of the blow-up are holomorphic. Which transition functions? The ones between the coordinate charts on $X \setminus \{x\}$ and $\tilde{V}$. We leave this check as an exercise.

Proposition 53. *The blow-up* $\mathrm{Bl}_x X$ *is a complex manifold and* $\pi : \mathrm{Bl}_x X \to X$ *is holomorphic.*

Notice that we made a choice of chart. Why are we justified in calling this *the* blow-up? Intuitively the idea is the following: we took an ideal of n equations defining the point x and turned it into a principal ideal defining the exceptional locus.

Lemma 54. *Let* $f = (f_1, \dots, f_n) : X \to \mathbb{C}^n$ *be a holomorphic map with* X *a connected complex manifold. Assume that* $f(X) \neq \{0\}$ *and that for all* $x \in X$ *such that* $f(x) = 0$ *the ideal* $(f_1, \dots, f_n) \subset \mathcal{O}_{X,x} = \mathcal{O}_n$ *is principal. Then there exists a unique holomorphic map* $\tilde{f} : X \to \mathrm{Bl}_0 \mathbb{C}^n$ *such that the diagram*

$$\begin{array}{ccc} & & \mathrm{Bl}_0 \mathbb{C}^n \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \pi \\ X & \xrightarrow{f} & \mathbb{C}^n \ni \{0\} \end{array}$$

commutes.

Proof. The uniqueness is straightforward. We know that $\tilde{f}(x) = f(x)$ for all $x \in f^{-1}(0) = \tilde{f}^{-1}(E)$. If we have another lift $\tilde{f}'$ then $\tilde{f} = \tilde{f}'$ on $X \setminus f^{-1}(0)$, which is a dense open set. But the set on which functions coincide is closed whence on all of X.

We may assume by uniqueness so we may assume that $X = U \subset \mathbb{C}^n$ with the point being the origin and that $f(0) = 0$. By hypothesis there exists $g \in \mathcal{O}_m$ of $(f_1, \dots, f_n) = (g) \subset \mathcal{O}_m$. By taking U very small we have $g = h_1 f_1 + \cdots + h_n f_n$ with $f_i = t_i g$ for some functions $h_i, t_i \in \mathcal{O}(U)$. Now we can write $g = h_1 t_1 g + \cdots + h_n t_n g$ and since the local ring is a domain we know that $1 = h_1 t_1 + \cdots + h_n t_n$. This means that for every x there exists some j such that $t_j(x) \neq 0$. Hence as points in $\mathbb{P}^{n-1}$, we have $(f_1(x) : \cdots : f_n(x)) = (t_1(x) : \cdots : t_n(x)$ and the key is that this makes sense for every x. Now we define $\tilde{f}$ as sending

$$x \mapsto ((f_1(x), \dots, f_n(x)), (t_1(x) : \cdots : t_n(x))).$$

□

Roughly speaking, given two different charts containing x, we get two blow-ups say $\mathrm{Bl}_0 V$ and $\mathrm{Bl}_0 V'$. One checks that the map induced $\mathrm{Bl}_0 V \to V \to V'$ principalizes, and hence we obtain a factorization through $\mathrm{Bl}_0 V'$. Doing the same argument the other way we obtain a map the other way. A nice reference for this is Hartshorne, something like II.7.14.

5. January 24, 2018

5.1. **Hypersurfaces are hard.** Claim: hypersurfaces are hard. Let's look at some open questions.

Problem: Let $F \in \mathbb{C}[x_0, \dots, x_d]$ be a homogeneous polynomial of degree d. If F defines a smooth cubic fourfold (i.e. $d = 3$ and the ambient space is $\mathbb{P}^5$) $Z(F)$. Can $Z(F)$ not be *rational*? What does rational mean? It means, roughly, that the variety can be parameterized. More precisely, in this case we are asking whether $Z(F)$ is birational to $\mathbb{P}^4$. Some are known to be rational, but there is not a single construction of one that is not. In fact, one of the most famous results in 20th century algebraic geometry is that a cubic 3-fold is *never* rational. There are a few different proofs and they are all spectacular.

Problem: Assume that $d \gg 0$. Is there a bound $d_0 = d_0(n)$ such that for all $d \geq d_0$ there are no nonconstant holomorphic maps $\mathbb{C} \to Z(F)$? This is a conjecture known as hyperbolicity of hypersurfaces of large degree. There is a similar question for maps to complement $\mathbb{C} \to \mathbb{P}^n \setminus Z(F)$. People expect that $d \sim 2n + 1$.

So the goal of this course is to learn enough complex geometry to solve this latter conjecture of Kobayashi.

5.2. **Vectorbundles.** Let $k = \mathbb{R}, \mathbb{C}$ and X be a topological space.

Definition 55. A k-**vector bundle** of rank r on X is a topological space V together with a continuous map $\pi : V \to X$ such that

(1) for each $x \in X$, $V_x = \pi^{-1}(x)$ is a k-vector space of dimension r,
(2) for each $x \in X$ there exists a neighborhood $x \in U \subset X$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times k^{\oplus r}$ and $\phi(V_x) = \{x\} \times k^{\oplus r}$ such that the induced map $V_x \xrightarrow{\phi} \{x\} \times k^{\oplus r} \to k^{\oplus r}$ is a linear isomorphism of vector spaces.

A morphism of vector bundles over X is a continuous map $f : V_1 \to V_2$ commuting with the projections to X such that f restricted to each fiber is linear of constant rank.[1]

Closed under composition?

Remark 56. We use the usual terminology in calling X the base space, V the total space, and $\phi : \pi^{-1}(U) \to U \times k^{\oplus r}$ a local trivialization of V.

Notice that if we have (U_i, ϕ_i) and (U_j, ϕ_j) two local trivializations we can define

$$(\mathrm{id}, g_{ih}) := \phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times k^{\oplus r} \to (U_i \cap U_j) \times k^{\oplus r}$$

whence we obtain an isomorphism of vector spaces

$$g_{ij}(x) : k^{\oplus r} \to k^{\oplus r}$$

for each $x \in U_i \in U_j$. We can consolidate this data into a continuous map $g_{ij} : U_i \cap U_j \to GL_r(k)$. We call these functions the transition functions for the vector bundle with respect to our given choice of local trivializations. These transition functions satisfy some nice properties: for all i, j, k we ahve

(1) $g_{ii} = \mathrm{id}$,
(2) $g_{ij} g_{jk} g_{ki} = \mathrm{id}$ on $U_i \cap U_j \cap U_k$.

Notice that the regularity of the transition functions (and the choice of field) allows us to define smooth and holomorphic vector bundles. Explicitly:

(1) for $k = \mathbb{R}$ and X a smooth manifold we have that $GL_n(\mathbb{R})$ is a real Lie group whence we say that V is smooth if there exist local trivializations such that the transition functions are smooth. In this case V is a smooth manifold and $\pi : V \to X$ is a smooth map.
(2) for $k = \mathbb{C}$ and X a complex manifold we have that $GL_n(\mathbb{C})$ is a complex Lie group whence we say that V is holomorphic if there exist local trivializations such that the transition functions are holomorphic. In this case V is a complex manifold and $\pi : V \to X$ is a holomorphic map.

One can in fact specify a vector bundle with only transition functions.

[1]This last condition is to ensure that kernels and cokernels are again vector bundles.

Lemma 57. *For X a topological space, the data of (U_i, g_{ij}) satisfying the properties above is equivalent to the data of a vector bundle V up to isomorphism. If X and the transition functions are smooth or holomorphic then so is V.*

Proof sketch. Let us outline the construction: the open cover gives us something trivial on each open, which we glue together. Define

$$\tilde{V} := \coprod_{i \in I} (U_i \times k^{\oplus r}).$$

There is an equivalence relation on $\tilde{V}$ where we identify

$$(x, v) \in U_i \times k^{\oplus r} \sim (y, w) \in U_j \times k^{\oplus r}$$

if and only if

$$x = y \quad v = g_{ij}(x) w.$$

Then take $V = \tilde{V}/\sim$, which one checks is a vector bundle with the requisite properties. □

Definition 58. A continuous, smooth, or holomorphic **section** of $\pi : V \to X$ over an open set $U \subset X$ is a continuous, smooth, or holomorphic map $s : U \to V$ such that

$$\pi \circ s = \mathrm{id}_U .$$

The sections over an open form a group that we denote $\Gamma(U, V)$. We will call the global sections the sections over all of X, $\Gamma(X, V)$.

Remark 59. To fix some terminology, a line bundle will mean a vector bundle of rank 1. Notice moreover that any linear algebraic construction producing new vector spaces from old ones yields similarly new vector bundles from old ones. The direct sum, for instance, sums transition functions while the tensor product takes products. We can also pass to the dual bundle, which has as its transition functions the transpose. Finally notice that the top exterior power, the determinant bundle, is always a line bundle.

Example 60 (Tautological bundle on $\mathbb{P}^n$)**.** Recall that we had constructed the blow-up $\mathrm{Bl}_0 \mathbb{C}^{n+1}$ which had two projections $\mathrm{Bl}_0 \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ and $q : \mathrm{Bl}_0 \mathbb{C}^{n+1} \to \mathbb{P}^n$. Intuitively we see that the latter should be a line bundle as we associate to every point the set of lines passing through that point. Indeed we claim that $q : Y \to \mathbb{P}^n$ is a holomorphic line bundle on $\mathbb{P}^n$. Recall that we had an open cover $V_i \cong q^{-1} U_i \to U_i \times \mathbb{C}$, with the isomorphism given $(z, \ell) \mapsto (\ell, z_i)$ which is exactly giving us a local trivialization. Clearly the transition functions are $g_{ij} = z_i / z_j$, which is holomorphic because $z_i, z_j \neq 0$ and we obtain a line bundle.

Exercise 61. Show that this tautological bundle has no global holomorphic sections.

6. January 26, 2018

Last time we talked about the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$, which happens to have total space $\mathrm{Bl}_0 \mathbb{C}^{n+1}$. Here is a nice argument to see why this bundle has no global sections (c.f. the exercise from last time). A global section would induce a global section $\mathbb{C}^{n+1} \times \mathbb{P}^n$ along the inclusion $\mathrm{Bl}_0 \mathbb{C}^{n+1} \subset \mathbb{C}^{n+1} \times \mathbb{P}^n$. But this latter bundle is trivial bundle over a compact space. The only global sections are the constant sections. But since it factors through the blowup (the point must be on every line through the origin) it must be the zero section.

Definition 62. For $m > 0$ we define $\mathcal{O}_{\mathbb{P}^n}(-m) := \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes m}$. Similarly we define $\mathcal{O}_{\mathbb{P}^n}(-1) := \mathcal{O}_{\mathbb{P}^n}(-1)^*$ and then $\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$. Finally we have the trivial bundle $\mathcal{O}_{\mathbb{P}^n}$.

It turns out, as we will see later, these are all the holomorphic line bundles on $\mathbb{P}^n$.

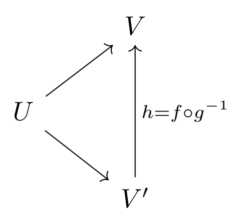
Exercise 63. Why does $\mathcal{O}_{\mathbb{P}^n}(-m)$ not admit any global sections? Both in the algebraic and holomorphic categories.

6.1. **(Co)tangent bundles.** Let's first consider the smooth category. Let X be a smooth manifold of dimension n and let $x \in X$ be a point. Let $f : U \subset X \to V \subset \mathbb{R}^n$ be a local chart, whose coordinate functions we write $f_i := x_i \circ f$, such that $f(x) = 0$. On V we have the usual vector fields, which we think of as derivations $\partial_{x_1}, \ldots, \partial_{x_n}$ of $\mathcal{A}_{\mathbb{R}^n}(V)$. These derivations also act on $\mathcal{A}_X(U)$ by

$$\phi \in \mathcal{A}_X(U) \mapsto (\partial_{x_i}\phi)(y) = \partial_{x_i}(\phi \circ f^{-1})(f(y))$$

for $y \in X$. These derivations form a vector space $T_0\mathbb{R}^n \cong \mathbb{R}^n$ associated to the point $x \in X \mapsto 0 \in V$. We expect these tangent spaces to fit together to yield a tangent bundle TX.

The smooth tangent bundle TX has fiber over $x \in X$ the vector space T_xX. We write the transition functions for this bundle as follows. Suppose we have



and $x_1, \ldots, x_n$ coordinates on V and $y_1, \ldots, y_n$ coordinates on V'. We want a formula for the change of charts for a vector field. Say we have a vector field on V,

$$\sum_{i=1}^n a_i(x)\partial x_i$$

but on V' it is written

$$\sum_{i=1}^n b_i(y)\partial_{y_i}.$$

Now for any smooth $\psi : V \to \mathbb{R}$, we look at the composition $V' \xrightarrow{h} V \xrightarrow{\psi} \mathbb{R}$:

$$\frac{\partial(\psi \circ h)}{\partial y_j}(h^{-1}(x)) = \sum_{i=1}^n \left(\frac{\partial h_i}{\partial y_j}(h^{-1}(x)) \right) \frac{\partial \psi}{\partial x_i}(x)$$

which is nothing but the usual multivariable chain rule. As vector field,s then, we have that

$$\partial_{y_j} = \sum_{i=1}^{n} \frac{\partial h_i}{\partial y_j}(h^{-1}(\cdot))\partial_{x_i}.$$

In other words, we have

$$a_j(x) = \sum_{i=1}^{n} \frac{\partial h_j}{\partial y_j}(h^{-1}(x))b_i(h^{-1}(x)).$$

Definition 64. Let X be a connected smooth manifold of dimension n over $\mathbb{R}$ with an atlas $(U_\alpha, f_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n)$ with coordinates x_i^α on V_α. Then we define

$$h_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}.$$

The **real tangent bundle of** X, denote $T_\mathbb{R} X$ is the smooth vector bundle of rank n given transition functions

$$g_{\alpha\beta} = \mathcal{J}(h_{\alpha\beta}) \circ f_\beta : U_\alpha \cap U_\beta \to GL_n(\mathbb{R}).$$

Here $\mathcal{J}(h_{\alpha\beta})$ is the Jacobian, the usual matrix of partials.

Let's now turn to the situation of interest to us, where X is a complex manifold. Fix $x \in X$ and take a coordinate chart $f : U \ni x \to V \subset \mathbb{C}^n$ sending $x \mapsto 0$. We can write coordinates on $\mathbb{C}^n$ as $z_1, \ldots, z_n$ or as real coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $V \subset \mathbb{R}^{2n}$. We have

$$T_{\mathbb{R},x} X = \mathbb{R}\langle \partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n} \rangle.$$

We now define the complexified real tangent bundle

$$\begin{aligned} T_{\mathbb{C},x} X := T_{\mathbb{R},x} X &\cong \mathbb{C}\langle \partial_{x_1}, \ldots, \partial_{y_n} \rangle \\ &\cong \langle \partial_{z_1}, \ldots, \partial_{z_n}, \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n} \\ &\cong \langle \partial_{z_1}, \ldots, \partial_{z_n} \rangle \oplus \langle \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n} \\ &:= T'_x X \oplus T''_x X \end{aligned}$$

which splits into the **holomorphic and antiholomorphic tangent spaces** of X. In this formalism we've produced the tangent space and its conjugate simultaneously. Indeed, inside $T_{\mathbb{C},x} X$ we have that $T''_x X = \overline{T'_x X}$ since $\partial_{\bar{z}_i}$ are conjugate to ∂_{z_i}.

Notice now that we have a map $T_{\mathbb{R},x} X \hookrightarrow T_{\mathbb{C},x} X$ sending $\partial_{x_i} \mapsto \partial_{z_i}$ and $\partial_{y_i} \mapsto \partial_{\bar{z}_i}$. We also have a map $T_{\mathbb{C},x} X \twoheadrightarrow T'_x X$ which is just the projection. The composition of these two maps, it is easy to see, is an isomorphism of $\mathbb{R}$-vector spaces.

Mihnea had a correction here? Something about $\partial/\partial y_j \mapsto i\partial/\partial z_j$?

Definition 65. We obtain the **holomorphic tangent bundle** $T'X$ of a complex manifold X whose transition functions are

$$g_{\alpha\beta} = \mathcal{J}(h_{\alpha\beta}) \circ f_\beta$$

where here we are taking the Jacobian with respect to the holomorphic variables z and $h_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$. We will call holomorphic sections of this bundle the holomorphic vector fields.

Let's say a few words about differentials about maps. Let's say that $f: X \to Y$ is a holomorphic map of complex manifolds. Say $x \mapsto y$ under f and we have coordinate charts $U \ni x$ and $V \ni y$ that map biholomorphically to $U' \in \mathbb{C}^n$ and $V' \in \mathbb{C}^m$. If we use coordinates z_i on U' and w_i on V', then we can write f as $(f_1, \ldots, f_m)$. There exists a real differential as usual $df_* : T_{\mathbb{R},x}X \to T_{\mathbb{R},y}Y$. This induces a map, after tensoring by $\mathbb{C}$, that we call

$$df_* : T_{\mathbb{C},x}X \to T_{\mathbb{C},y}Y.$$

Check whether this i should actually be there.

There is a formula

$$df_*(\partial_{z_j}) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j}\partial_{w_k} + i\frac{\partial \bar{f}_k}{\partial z_j}\partial_{\bar{w}_k}.$$

This formula applies to all smooth maps f, but if f is holomorphic then $\partial f_k/\partial \bar{z}_j = 0$ for all k, j whence the formula reduces to

$$df_*(\partial_{z_j}) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j}\partial_{w_k},$$

which is the exact same formula as in the real setting. Notice this is just the holomorphic Jacobian.

Exercise 66. For the real Jacobian $\mathcal{J}_{\mathbb{R}}(f)$ we have that

$$\det \mathcal{J}_{\mathbb{R}}(f) = |\det \mathcal{J}(f)|^2.$$

In fact one can check that we have

$$\det \mathcal{J}_{\mathbb{R}}(f) = \begin{pmatrix} \mathcal{J}(f) & 0 \\ 0 & \overline{\mathcal{J}(f)} \end{pmatrix}.$$

It follows that complex manifolds are always orientable!

7. January 29, 2018

Theorem 67 (Implicit function theorem)**.** *Let $f : X \to Y$ be a holomorphic map of complex manifolds. Assume that $df_x : T_xX \to T_yY$ has constant rank r for all $x \in X$. Then for all $y \in Y$ the set $f^{-1}(y)$ is either empty or a complex submanifold of dimension $\dim X - r$ of X.*

Notice that we can talk about analytic subsets of complex manifolds. They are locally given as the vanishing locus of a finite set of holomorphic functions. Now suppose $f_1, \dots, f_k$ define $Z \subset X$ locally. Then we can look at the matrix of partials

$$\frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_k)}$$

has constant rank k then Z is a submanifold of X of dimension $\dim X - k$. This is a special case of the theorem above.

Definition 68. If $Z \subset X$ is an analytic subset then we say that $x \in Z$ is a **smooth** or **analytic** point if Z is a submanifold around x. Otherwise x is **singular**.

Example 69. The typical example is something like $z^2 = w^2 + w^3$. Differentiating with respect to z we have $2z$ and with respect to w we have $2w + 3w^2$. This shows that there is a singular point at the origin, as we expect from the picture.

Exercise 70. The singular locus of Z is a proper analytic subset of X.

7.1. **Forms on smooth manifolds.** Let's recall the real smooth case to set notation. A **smooth k-form** ω on X is a global section of $\Lambda^k T^*_{\mathbb{R}} X$. In other words, this is the data of smoothly varying elements $\omega(x) \in \Lambda^k T^*_{\mathbb{R},x} X$. In particular ω gives us a multilinear alternating function on smooth vector fields, $(v_1, \dots, v_k) \mapsto \omega(v_1, \dots, v_k)$. Locally in coordinates x_i the vector fields are spanned by $\partial_{x_1}, \dots, \partial_{x_n}$. This yields a standard dual basis dx^i of 1-forms such that $dx^i(\partial_{x_i}) = \delta^i_j$. Let us denote

$$\mathcal{A}^k(X) = \Gamma(X, \Lambda^k T^*_{\mathbb{R}} X).$$

Now any 1-form can be written

$$\omega = \phi_1 dx^1 + \cdots \phi_n dx^n$$

for $\phi_i \in \mathcal{A}(U)$. It follows that any k-form can be written

$$\omega = \sum_{i_1 < \cdots < i_k} \phi_{i_1, \dots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{|I|=k} \phi_I(x) dx^I$$

where we are using multi-index notation. Notice that by acting on vectors we can write

$$\phi_I(x) = \omega(x)(\partial_{x_1}, \dots, \partial_{x_k}).$$

Locally differential forms are not so interesting cohomologically, but globally we obtain de Rham cohomology.

Definition 71. Let $\omega \in \mathcal{A}^k(X)$. There is an **exterior derivative** assigning to ω a $(k+1)$-form $d\omega$, given locally,

$$d\omega = \sum_{j=1}^{n} \sum_{|I|=k} \frac{\partial \phi_I}{\partial x_j} dx^j \wedge dx^I.$$

This yields a map

$$d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$$

satisfying the Leibniz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{|\omega_1||\omega_2|}\omega_1 \wedge d\omega_2$$

and such that

$$d^2 = 0.$$

This definition yields the **de Rham complex** of X:

$$A^\bullet X := \quad \mathcal{A}^0 X \to \mathcal{A}^1 X \to \cdots \to \mathcal{A}^n X.$$

This sequence is certainly not exact since globally not every closed form is exact.

Definition 72. We define the **de Rham cohomology groups** to be the cohomology of $A^\bullet X$, i.e.

$$H^i_{\mathsf{dR}} X := \frac{\ker(d : \mathcal{A}^i X \to \mathcal{A}^{i+1} X}{d : \mathcal{A}^{i-1} \to \mathcal{A}^i X}.$$

Theorem 73 (de Rham)**.** *For X any smooth manifold we have*

$$H^i(X; \mathbb{R}) \cong H^i_{\mathsf{dR}}(X)$$

for all i.

Recall that we can pullback forms between manifolds. In particular given a smooth map $f : X \to Y$ we obtain a linear map $f^* : \mathcal{A}^k Y \to \mathcal{A}^k X$. Locally, in coordinates x_i on X and y_i on Y writing $f_i = y_i \circ f(x_1, \ldots, x_n)$, we define

$$f^* dy^i := df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx^j.$$

7.2. **Forms on complex manifolds.** Let X be a complex manifold of dimension n and suppose we have local coordinates $z_j = x_j + iy_j$. We define

$$dz^j := dx^j + idy^j \qquad d\bar{z}^j := dx^j - idy^j.$$

These objects live in the complexified cotangent bundle of X. Notice now that any $\omega \in \mathcal{A}^k X$ can be written as

$$\omega = \sum_{|I|+|J|=k} \phi_{IJ} dz^I \wedge d\bar{z}^J$$

for smooth complex-valued functions ϕ_{IJ}. For instance one can compute that $dz \wedge d\bar{z} = -2idx \wedge dy$.

Definition 74. We say that $\omega \in \mathcal{A}^k X$ is of type (p, q) if locally it can be written as

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \phi_{IJ} dz^I \wedge d\bar{z}^J.$$

One checks easily that this condition globalizes. Hence we obtain a linear subspace $\mathcal{A}^{p,q} X \subset \mathcal{A}^k X$.

Let's consider now how d acts with respect to this notion. Well clearly since it takes derivatives and tacks on 1-forms, $d : \mathcal{A}^k X \to \mathcal{A}^{k+1} X$ cannot possible reduce p or q. Hence, since d has to increase $p+q$ by 1, we obtain a decomposition

$$d = \partial + \bar{\partial} : \mathcal{A}^{p,q} X \to \mathcal{A}^{p+1,q} X \oplus \mathcal{A}^{p,q} X.$$

Let's write one of them locally:

$$\bar{\partial}\left(\sum_{IJ} \phi_{IJ} dz^I \wedge d\bar{z}^J\right) = \sum_{kIJ} \frac{\partial \phi_{IJ}}{\partial \bar{z}_k} d\bar{z}^k \wedge dz^I \wedge dz^J.$$

Indeed we will be focusing mostly on $\bar{\partial}$ instead of ∂.

Exercise 75. Check that $\bar{\partial}^2 = 0$.

Try to turn in the first homework by the end of this week or the beginning of next week.

Let's meet again tomorrow at 2pm in Lunt 105 for the last makeup session.

8. January 30, 2018

8.1. **Dolbeault cohomology.** Recall that last time we defined $\mathcal{A}^{p,q}X \subset \mathcal{A}^k X$ where $p+q=k$. We saw that the exterior derivative split as

$$d = \partial + \bar{\partial}$$

Definition 76. The **Dolbeault complexes of** X are the complexes (for all p)

$$\mathcal{A}^{p,0}X \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n}X$$

The (p,q)-Dolbeault cohomology group is

$$H^{p,q}X := \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1})}{\operatorname{im}(\bar{\partial} : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q})}.$$

Notice that based on p this complex will vary in length. These complexes give rise to a refinement of the de Rham cohomology.

Example 77. Notice that $H^{p,0}X = \{\omega \in \mathcal{A}^{p,0}X \mid \bar{\partial}\omega = 0\}$ is precisely the space of holomorphic p-forms.

Exercise 78. Check that $H^{1,0}\,\mathbb{P}^n = 0$.

We are interested in relating Dolbeault cohomology to sheaf cohomology, as the latter allows some useful computation techniques. The following result is the first step in this direction. It is an analog of the usual Poincaré lemma which says that smooth d-closed forms and locally d-exact.

Theorem 79 ($\bar{\partial}$-Poincaré lemma)**.** *Let $\omega \in \mathcal{A}^{p,q+1}U$ for $U \subset \mathbb{C}^n$ open, with $q \geq 0$ such that $\bar{\partial}\omega = 0$, i.e. ω is ($\bar{\partial}$-)closed. Then for every relatively compact subset $K \subset U$ (i.e. $\bar{K}$ is compact) there exists $\psi \in \mathcal{A}^{p,q}K$ such that $\bar{\partial}\psi = \omega$.*

Proof. See, for instance, Huybrechts section 1.3. □

Corollary 80. *Let $\mathcal{D} = \{z \in \mathbb{C}^n \mid |z_j| < r_j \forall j\}$ for some r_j. Then*

$$H^{p,q}D = 0 \quad q \geq 1.$$

Moreover we may assume that (some of) the $r_j = \infty$.

Proof. See for instance Griffiths-Harris pp.25-27. The idea is to cover the space with relatively compact sets getting bigger and bigger and applying the lemma above. Of course, it does not follow trivially like that: the expressions of exactness one obtains on each set may agree only up to closed forms, so there is some work to be done. □

8.2. **Integration.** Over the next few lectures we will define and discuss basic properties of Kähler manifolds. We start with some calculus of real manifolds. Let X be a smooth manifold of dimension n (over $\mathbb{R}$). Recall that, roughly speaking, X is orientable if we can make a consistent choice of basis in each $T_{\mathbb{R},x}X$. More precisely, given a vector space V we can put an equivalence relation on the set of bases on V. The equivalence classes are given by the sign of the change of basis matrix. This globalizes easily to manifolds: if (U_i, ϕ_i) is an atlas with $g_{ij=\phi_i \circ \phi_j^{-1}}$ transition functions. Then X is orientable if and only if g_{ij} are orientation preserving, i.e. $\det \mathcal{J}_{\mathbb{R}}(g_{ij}) > 0$.

The notion of orientation is useful in discussing integration. Let $\omega \in \mathcal{A}^n X$ have **compact support**, i.e. if

$$\operatorname{supp}\omega := \{x \mid \omega(x) \neq 0\},$$

then $\bar{K}$ compact. Take a finite subcover of charts U_i such that $K \subset U_i$. We choose a partition of unity $\{\phi_i\}$ associated to U_i. Recall that these functions have compact support in the U_i and they sum to 1 at each point. The charts identify $U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n$ and we obtain

$$\rho_i \omega = f_i dx_1^i \wedge \cdots \wedge dx_n^i,$$

for some f_i compactly supported $V_i \to \mathbb{R}$. Finally we can define

$$\int_X \omega := \sum_i \int_{V_i} f_i d\mu,$$

where μ is the usual Lebesgue measure on V_i. We now need to check that this is well-defined regardless of our choice of partition of unity and coordinate charts.

Indeed if we have another set of coordinates $x_1^j, \ldots, x_n^j$ then

$$dx_1^i \wedge \cdots \wedge dx_n^i = \det \mathcal{J}_{\mathbb{R}}(g_{ij}) g_{ij}^{-1} dx_1^j \wedge \cdots \wedge dx_n^j.$$

It now follows from orientability and the change of variables formula that these yield the same integral.

Theorem 81 (Stokes)**.** *If $\eta \in \mathcal{A}^{n-1}X$ has compact support then*

$$\int_X d\eta = 0.$$

Notice that we have a mapping $\mathcal{A}^n X \to \mathbb{R}$ sending $\omega \mapsto \int_X \omega$ which by Stokes' theorem descends to a map on de Rham cohomology. Indeed, for X compact the map on cohomology is an isomorphism.

Now let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Let (U_i, ϕ_i) be an atlas and g_{ij} be the transition functions. We saw earlier that

$$\det \mathcal{J}_{\mathbb{R}}(g_{ij}) = |\det \mathcal{J}(g_{ij})|^2 > 0$$

whence every complex manifold is orientable.

Choose local coordinates $z_j = x_j + iy_j$ and fix an orientation $x_1, y_1, \ldots, x_n, y_n$. We integrate with respect to

$$(dz^1 \wedge d\bar{z}^1) \wedge \cdots \wedge (dz^n \wedge d\bar{z}^n)$$

which will play the role of the Lebesgue measure above. One can compute that

$$(dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n) = \frac{i^n}{2^n}(dz^1 \wedge d\bar{z}^1) \wedge \cdots \wedge (dz^n \wedge d\bar{z}^n).$$

In particular given $\omega \in \mathcal{A}^{n,n}X = \mathcal{A}^{2n}X$ of compact support, we now have, with respect to this new measure,

$$\int_X \omega \in \mathbb{C}.$$

8.3. **Hermitian metrics.** For X a smooth manifold of dimension n, a **Riemannian metric** on X is a collection of positive-definite symmetric bilinear forms $g_x : T_{\mathbb{R},x}X \times T_{\mathbb{R},x} \to \mathbb{R}$ varying smoothly with x. More precisely, for $U \subset X$ open and u, v any two smooth vector fields on U then we require that $g(u, v) : U \to \mathbb{R}$ be smooth.

Recall that $g : V \times V \to \mathbb{R}$ is positive definite if for all nonzero v we have that $g(v, v) > 0$ and symmetric means that $g(u, v) = g(v, u)$. The usual obvious example is the usual Euclidean inner product on $\mathbb{R}^n$. If we choose a basis we can represent g by a matrix, call it A, and the positivity yields $v^t A v > 0$ for all $v \neq 0$ and the symmetry simply yields that A is symmetric.

If $x_1, \dots, x_n$ are local coordinates on U we can write

$$g_{ij}(x) := g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) : U \to \mathbb{R}.$$

We might write the matrix as $G(x) = (g_{ij}(x))_{1 \leq i,j \leq n}$. The Euclidean metric on $\mathbb{R}^n$, for instance, is written in the usual coordinates as $g_{ij}(x) = \delta_{ij}$. Notice that given an embedded submanifold $X \subset \mathbb{R}^n$ we can induce a metric on X using the Euclidean metric.

9. January 31, 2018

9.1. **Hermitian metrics (continued).** Let V be a finite dimensional complex vector space with $\dim_{\mathbb{C}} V = n$. Denote by $J : V \to V$ the scalar multiplication by i, which exists because V is complex. Notice that $J^2 = -\operatorname{id}_V$.

Definition 82. A **Hermitian form** on V is a mapping $h : V \times V \to \mathbb{C}$ such that

(1) h is $\mathbb{C}$-linear in the first entry,
(2) $h(v, w) = \overline{h(w, v)}$.

Note that $h(v, v) \in \mathbb{R}$ for any $v \in V$. We say that h is **positive** if $h(v, v) > 0$ for each $v \neq 0$.

For any Hermitian form h we have associated

$$\begin{aligned} g :& V \times V \to \mathbb{R} & \omega :& & V \times V \to \mathbb{R} \\ g(v, w) =& \Re h(v, w) & \omega(v, w) & & = -\Im h(v, w). \end{aligned}$$

Lemma 83. *Let h be a positive definite Hermitian form on V. Then*

(1) g is an inner product (symmetric positive-definite bilinear form) on V
(2) ω is a (real) alternating bilinear form on V.

Proof. Notice that

$$g(v, w) = \frac{1}{2}(h(v, w) + \overline{h(v, w)}) = \frac{1}{2}(h(v, w) + h(w, v))$$

which is obviously symmetric bilinear. Positivity follows from the positivity of h.

For the alternating property of ω we simply write

$$\omega(v, w) = \frac{1}{2}\left(h(v, w) - h(w, v)\right).$$

□

Remark 84. We can recover h from g. Indeed, just use

$$h(v, w) = g(v, w) + ig(v, Jw)$$

and the formula for g in terms of h.

For an arbitrary inner product g, however, it is not guaranteed that this expression is Hermitian. Notice that

$$\overline{h(w, v)} = g(w, v) - ig(w, Jv)$$

so we must have, for h to be Hermitian, that

$$g(v, Jw) + g(JV, w) = 0,$$

or equivalently

$$g(v, w) = g(Jv, Jw).$$

In this case we say that g is **compatible** with J.

Instead we might start with ω and obtain h by taking $g(v, w) = \omega(v, Jw)$. Again the resulting h will be Hermitian if and only if $\omega(v, w) = \omega(Jv, Jw)$.

To summarize: any of the objects J, g, ω determines the other two. Let's globalize these definitions to the case of manifolds.

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Recall that we have for every $x \in X$, mappings

$$T_{\mathbb{R},x}X \hookrightarrow T_{\mathbb{C},x} \twoheadrightarrow T'_x X$$

sending

$$\partial_{x_j} \mapsto \partial_{z_j} \quad \text{and} \quad \partial_{y_j} \mapsto i\partial_{z_j}.$$

Definition 85. A **Hermitian metric** on X is a collection of Hermitian forms $h_x : T'X \times T'X \to \mathbb{C}$ on the holomorphic tangent spaces for each $x \in X$ such that the associated forms g_x give a Riemannian metric on X.

Remark 86. If h is a Hermitian metric on X we obtain automatically two pieces of data. The first is of course the Riemannian metric g, which is the real part of h. The second datum is the real two-form $\omega \in \mathcal{A}^2(X)$ induced by the (negative of the) imaginary part of h.

Let's look at the local expressions for g and ω. Write $z_j = x_j + iy_j$ for local coordinates on some chart on X. Applying h to the induced holomorphic vector fields we obtain a matrix at each point

$$H = (h_{kl})_{1\le k,l\le n}, \qquad h_{kl} = h(\partial_{z_k}, \partial_{z_l}).$$

It is easy to check that H is a positive-definite Hermitian matrix, i.e. $H = H^\dagger$ at each point. Notice that the matrix acts as a bilinear form via

$$(v, w) \mapsto {}^t v \cdot H \cdot \bar{w}.$$

In particular ${}^t vH\bar{v} > 0$ for all $v \neq 0$.

We can now write

$$g(\partial_{x_k}, \partial_{x_l}) = \Re h(\partial_{z_k}, \partial_{z_l}) = \Re h_{kl}.$$

Similarly

$$g(\partial_{x_k}, \partial_{y_l}) = \Re h(\partial_{z_k}, i\partial_{z_l}) = \Im h_{kl}.$$

Hence we can write the matrix of the Riemannian metric as

$$G = \begin{pmatrix} \Re h & \Im h \\ -\Im h & \Re h \end{pmatrix}.$$

We should of course obtain a symmetric matrix. Notice that $H = \Re H + i\Im H$ so ${}^t H = (\Re H)^t + i(\Im H)^t$. Conjugating, we find that

$$\overline{{}^t H} = (\Re H)^t - i(\Im H)^t$$

whence H is Hermitian if and only if $\Re H$ is symmetric and $\Im H = -(\Im H)^t$.

What about ω? Let's express it in terms of $dz, d\bar{z}$. We compute

$$\begin{aligned} \omega(\partial_{x_k}, \partial_{x_l}) &= -\Im h(\partial_{z_k}, \partial_{z_l}) = -\Im h_{kl} \\ \omega(\partial_{x_k}, \partial_{y_l)} &= -\Im h(\partial_{z_k}, id_{z_l} = \Re h_{kl}, \end{aligned}$$

and so on. What we're really after is, for instance,

$$\begin{aligned} 4\omega(\partial_{z_k}, \partial_{\bar{z}_l}) &= \omega(\partial_{x_k} - i\partial_{y_k}, \partial_{x_l} + i\partial_{y_l}) \\ &= -\Im h_{kl} + i\Re h_{kl} + i\Re h_{kl} - \Im h_{kl} \\ &= -2ih_{kl}. \end{aligned}$$

Similarly one computes

$$\omega(\partial_{z_k}, \partial_{z_l}) = 0 \qquad \omega(\partial_{\bar{z}_k}, \partial_{\bar{z}_l}),$$

from which it follows that we can write, locally,

$$\omega = \frac{i}{2}\sum_{k,l}^{n} h_{kl} dz^k \wedge d\bar{z}^l.$$

In other words ω is a $(1,1)$-form. We call this form **positive** as the matrix of its coefficients is positive-definite by positivity of h.

Definition 87. A Hermitian metric h is called **Kähler** if $d\omega = 0$. A complex manifold X is called **Kähler** if it admits a Kähler metric.

The most obvious example is $\mathbb{C}^n$ equipped with the Euclidean metric. We take $h_{kl} = \delta_{kl}$, the identity matrix. The associated two-form is just

$$\frac{i}{2}\sum_{k} dz^k \wedge d\bar{z}^k = \sum_{k} dx^k \wedge dy^k.$$

Next time we will see some more interesting examples.

10. February 2, 2018

Recall last time we discussed the notion of a Hermitian metric and how it gives rise to a Riemannian metric and a real two-form. This real two-form sits, in the complexification of the real tangent bundle, as a (1,1)-form. We said that a Hermitian metric was Kähler if this form is in fact closed.

Remark 88. Notice that given two Kähler forms we can add them, or we can scale them positively. Indeed, we obtain a cone.

Example 89. Here are a few basic examples.

(1) There is the obvious Kähler metric on $\mathbb{C}^n$ where the form is written $\sum_{k=1}^n dx^k \wedge dy^k$. Notice that the metric is invariant under the translation action whence we obtain Kähler forms on the torus $\mathbb{C}^n/\Lambda$.
(2) If X is a Riemann surface, since Hermitian metrics exist on any complex manifold, we can choose a Hermitian metric. The associated two-form is closed for dimensional reasons, whence every Hermitian metric on a Riemann surface is Kähler.
(3) Given a Hermitian metric h on X we can restrict it to a submanifold $Y \subset X$. Clearly the associated two-form on Y is just the pullback of the associated two-form on X. This implies that every submanifold of a Kähler manifold is Kähler.
(4) The projective space $\mathbb{P}^n$ is Kähler in a natural way via the Fubini-Study metric. We explain this below.

Proposition 90. *The projective space* $\mathbb{P}^n$ *is Kähler.*

Proof. Recall the atlas we have on $\mathbb{P}^n$, which we denote (U_i, ϕ_i), where

$$\phi_i(z_1 : \ldots : z_n) = (z_0/z_i, \ldots, \hat{i}, \ldots z_n/z_i)$$

which makes sense because $z_i \neq 0$. Let us denote $w_j = z_j/z_i$. Then over U_i define

$$\begin{aligned}\omega_i &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^n |w_k|^2 + 1\right)\\ &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=0}^n |\frac{z_k}{z_i}|^2\right) \in \mathcal{A}^{1,1}U_i.\end{aligned}$$

We claim that these ω_i patch together to a global form $\mathcal{A}^{1,1}\,\mathbb{P}^n$, i.e. $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$:

$$\begin{aligned}\log\left(\sum_{k=0}^n |\frac{z_k}{z_i}|^2\right) &= \log\left(|\frac{z_j}{z_i}|^2 \sum_{k=0}^n |\frac{z_k}{z_j}|^2\right)\\ &= \log\left(|\frac{z_j}{z_i}|^2\right) + \log\left(\sum_{k=0}^n |\frac{z_k}{z_j}|^2\right).\end{aligned}$$

Now applying $\partial\bar{\partial}$ to this extra term we obtain zero – indeed, check that $\partial\bar{\partial}\log|z|^2 = 0$. We write this form as

$$\omega_{\mathsf{FS}} \in \mathcal{A}^{1,1}\,\mathbb{P}^n\,.$$

This is only true for real-valued functions?

The two-form ω_{FS} is real simply because,

$$\overline{\partial\bar{\partial}} = \bar{\partial}\partial = -\partial\bar{\partial}$$

due to the fact that $d^2 = (\partial + \bar{\partial})^2 = 0$ and $\partial^2 = \bar{\partial}^2 = 0$. These relations also immediately imply that $d\omega = 0$. Finally we check that ω_{FS} is positive. A straightforward local computation shows that

$$\omega_i = \frac{i}{2\pi} \sum_{i,j=1}^{n} h'_{ij} \frac{dw^i \wedge d\bar{w}_j}{(\sum_{i=1}^n |w_i|^2 + 1)^2}$$

where

$$h'_{ij} = (\sum_{i=1}^{n} |w_i|^2 + 1) \cdot \delta_{ij} - \bar{w}_i w_j.$$

One can now do some work and show that the matrix of coefficients is positive-definite.

There is a slicker approach, however. Denote by $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the quotient map. We claim that ω_{FS} is the unique (1,1)-form on $\mathbb{P}^n$ such that

$$q^* \omega_{\mathsf{FS}} = \frac{i}{2\pi} \partial \bar{\partial} \log(|z_0|^2 + \cdots + |z_n|^2).$$

To see this, just check that it holds in every chart and that it is invariant under the scaling group action. Now notice that the unitary group $U(n+1)$ acts on $\mathbb{P}^n$. Given $A \in U(n+1)$, we act A on $[x]$ as $[Ax]$. The fact that this action is transitive is essentially coming from Gram-Schmidt process. We have that

$$\phi_A^* \omega_{\mathsf{FS}} = \omega_{\mathsf{FS}}$$

because all that appears in the definition of the form is the size $|z|$ which is preserved by the unitary action. Now, by transitivity and left-invariance, it is enough to verify positivity at $P = (1 : 0 : \cdots : 0)$, say. In the local form given above one checks that

$$\omega_{\mathsf{FS}}(P) = \frac{i}{2\pi} \sum_i dw^i \wedge d\bar{w}^i$$

so the matrix is (maybe up to some constants missing) just the identity matrix. □

Corollary 91. *Every submanifold of $\mathbb{P}^n$ is Kähler.*

Proof. See the third example above. □

In particular, all smooth projective varieties are Kähler.

Remark 92. Notice that $[\omega_{\mathsf{FS}}] \in H^2(\mathbb{P}^n, \mathbb{R}) \cong \mathbb{R}$. But in fact we will see later that $[\omega_{\mathsf{FS}}] \in H^2(\mathbb{P}^n, \mathbb{Z})$ (it will be the first Chern class of $\mathcal{O}(1)$). Moreover, since it is a (1,1)-form, we have that $[\omega_{\mathsf{FS}}] \in H^{1,1}\,\mathbb{P}^n \cong \mathbb{C}$. Indeed, $[\omega_{\mathsf{FS}}]$ is a generator for these groups.

We leave it as a homework exercise, for instance, to check that

$$\int_{\mathbb{P}^n} \omega_{\mathsf{FS}}^n = 1,$$

at least for $n = 1$.

Next time we will discuss volumes of Kähler manifolds and Wirtinger's theorem.

11. February 5, 2018

11.1. **Volumes.** Let V be a $\mathbb{R}$-vector space with $\dim_{\mathbb{R}} V = n$. Fix an orientation of V. We say that $v_1 \wedge \cdots \wedge v_n \in \Lambda^n V$ is **positive** if the collection of vectors $v_1, \ldots, v_n$ forms a positive basis. Now we fix an inner product $g : V \times V \to \mathbb{R}$, which induces an inner product on any $\Lambda^k V$, denote it by abuse of notation $g : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$ given on pure wedges (and extend by linearity) as

$$g(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det g(v_i, w_j).$$

Note that in $\Lambda^n V \cong \mathbb{R}$ there exists a unique element ω such that ω is positive and $g(\omega, \omega) = 1$. We call ω a **fundamental element**. Of course, $\omega = u_1 \wedge \cdots \wedge u_n$ for any positive orthonormal basis $u_1, \ldots, u_n$.

As usual, g induces an isomorphism $V \to V^*$ by sending $v \mapsto g(v, -)$. Any structure on V can now be induced on V^* by this isomorphism, e.g. V^* is naturally oriented, an inner product space, etc. In particular V^* has a corresponding fundamental element $\omega^* \in \Lambda^n V^*$.

Definition 93. Let X be an oriented smooth manifold of real dimension n with Riemannian metric g. The **volume form** of (X, g) is the unique form

$$\text{vol}(g) \in \mathcal{A}^n X$$

such that $\text{vol}(g)(x) = \omega_x \in \Lambda^n T_x^* X$ is the fundamental element for each $x \in X$.

The local expression, in coordinates $x_1, \ldots, x_n$, for the volume form is

$$\text{vol}(g) = \sqrt{G(x)} dx^1 \wedge \cdots \wedge dx^n \qquad G(x) = \det(g_{ij}(x)).$$

The square root term is coming from our normalization in the definition of the fundamental element.

Definition 94. The **volume of** X is

$$\text{vol}(X) := \int_X \text{vol}(g).$$

Example 95. If $X = S_r^2 \subset \mathbb{R}^3$ with the metric induced from the Euclidean metric, then

$$\text{vol}(S_r^2) = 4\pi r^2,$$

the usual expression for the surface area of a sphere.

Now suppose (X, h) is a Hermitian complex manifold and g is the underlying Riemannian metric associated to h. To simplify computations, we choose a unitary basis $\partial_1, \ldots, \partial_n \in T_x' X$ for h at each point x (in a chart). Pick $\alpha_1, \ldots, \alpha_n \in T_x^{1,0} X$ a dual basis in the sense that $\alpha_i(\partial_j) = \delta_{ij}$. Recall the associated (1,1)-form of h which is in this unitary frame written

$$\omega = \frac{i}{2} \sum_{k=1}^n \alpha_k \wedge \bar{\alpha}_k.$$

Theorem 96 (Wirtinger)**.** *The volume form is given*

$$\text{vol}(g) = \frac{\omega^n}{n!}.$$

Proof. We simply compute locally:

$$\begin{aligned}\omega^n &= \frac{i^n}{2^n}(\sum_{k=1}^{n}\alpha_k\wedge\bar{\alpha}_k)^n \\ &= \frac{i^n}{2^n}\cdot n!\alpha_1\wedge\bar{\alpha}_1\wedge\cdots\wedge\alpha_n\wedge\bar{\alpha}_n.\end{aligned}$$

Recall now that the matrix for g is written

$$G=\begin{pmatrix}\Re H & \Im H\\ -\Im H & \Re H\end{pmatrix}$$

whence (in our coordinates H is the identity matrix) we have

$$\begin{aligned}\mathrm{vol}(g) &= \sqrt{G(x)}dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n \\ &= \frac{i^n}{2^n}\alpha_1\wedge\bar{\alpha}_1\wedge\cdots\alpha_n\wedge\bar{\alpha}_n.\end{aligned}$$

□

Corollary 97. *The volume of a Hermitian complex manifold X is computed*

$$\mathrm{vol}(X)=\frac{1}{n!}\int_X\omega^n.$$

More generally for any submanifold $\iota: Y\subset X$ with $\dim_{\mathbb{C}} Y = p$ we have

$$\mathrm{vol}(Y)=\frac{1}{p!}\int_Y\iota^*\omega^p.$$

Thus there is a single form on X which is used to copmute volumes of all submanifolds of X. This is certainly not true for smooth manifolds! See, for instance, Griffiths and Harris for an example.

Example 98. Let $X=\mathbb{C}^n/\Lambda$ be a complex torus with the metric induced by the Euclidean metric on $\mathbb{C}^n$. Let's look at $n=1$: if D is the fundamental domain for the torus in $\mathbb{C}$, then we can compute

$$\mathrm{vol}(X)=\int_X\mathrm{vol}(g)=\int_D\mathrm{vol}(g)=\int_D d\mu=\mathrm{vol}(D)$$

where $d\mu$ is just the Lebesgue measure.

11.2. **First properties of Kähler manifolds.** Finally we come to something of substance.

Corollary 99. *If X is a compact Kähler manifold then the relevant even Betti numbers are non-zero i.e.*

$$b_{2k}(X)=\dim_{\mathbb{R}} H^{2k}(X;\mathbb{R})\neq 0$$

for $k=0,\ldots,n$.

Remark 100. This shouldn't be a mystery – in the algebraic case an n-dimensional variety always has algebraic subvarieties of any dimension (easy result in commutative algebra, say). So give yourself a subvariety $S\subset X\subset\mathbb{P}^n$. This will give us $[S]\in H_{2p}(X;\mathbb{Z})$, which by Poincaré duality yields a nontrivial class in $H^{2n-2p}(X;\mathbb{Z})$. This is very different than the noncompact complex-analytic setting – there may, for example, be surfaces with no curves on them.

Proof. Let ω be the associated $(1,1)$-form. Now since $d\omega = 0$ by the Leibniz rule we have that $d(\omega^k) = 0$ giving us a potentially nontrivial class. Indeed we claim that ω^k is not exact. If it were, then $\omega^k = d\eta$. Then

$$\omega^n = \omega^{n-k} \wedge \omega^k = \omega^{n-k} \wedge d^\eta = d(\omega^{n-k} \wedge \eta).$$

Now since X is compact, we apply Wirtinger's and Stokes' theorem

$$0 < \mathrm{vol}(X) = \frac{1}{n!} \int_X \omega^n = \frac{1}{n!} \int_X d(\omega^{n-k} \wedge \eta) = 0$$

we obtain a contradiction. □

Corollary 101. *If $Y \subset X$ is a compact complex p-dimensional submanifold of a Kähler manifold then Y is not the image of a boundary in X.*

Proof. If not then $Y = \phi(\partial M)$ for some map $\phi : M \to X$ and a manifold $(M, \partial M)$ with boundary. Now we imitate the proof above,

$$0 < \mathrm{vol}(Y) = \frac{1}{p!} \int_Y \iota^* \omega^p = \frac{1}{p!} \int_{\partial M} \phi^* \omega^p = \frac{1}{p!} \int_M d(\phi^* \omega^p) = 0,$$

which is a contradiction. □

12. February 7, 2018

12.1. More of Kähler manifolds.

Example 102 (Hopf surface)**.** Recall that the Hopf surface was constructed as a quotient by the action

$$\mathbb{Z} \times (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{C}^2 \setminus \{0\}$$

that sends, for a fixed $0 < \lambda < 1$,

$$(k, (z_1, z_2)) \mapsto (\lambda^k z, \lambda^k z_2).$$

Write $X = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ for the resulting compact complex surface. On the first homework I asked you to show that $X \cong S^3 \times S^1$ so I'm sure you already know how to do this. I'll still give you a hint. There is a diffeomorphism $S^3 \times \mathbb{R} \to \mathbb{C}^2 \setminus \{0\}$, where we think of $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, given as $((z_1, z_2), t) \mapsto (\lambda^t z_1, \lambda^t z_2)$. Then there is a further action of $\mathbb{Z}$ adding an extra $k \in \mathbb{Z}$ in the exponent of λ. Passing to the quotient we obtain $S^3 \times S^1$.

We know the cohomology groups of spheres quite well. In particular $H^p(S^n; \mathbb{Z})$ is $\mathbb{Z}$ for $p = 0, n$ and zero other wise. To compute the Betti numbers of X (the ranks of the real cohomology groups), then, we use the Kunneth formula (working over a field):

$$H^i(X; \mathbb{R}) \cong \oplus_{p+q=i} H^p(S^3; \mathbb{R}) \otimes H^q(S^1; \mathbb{R}).$$

We then have

$$b_0(X) = b_1(X) = b_3(X) = b_4(X) = 1 \quad \text{and} \quad b_2(X) = 0.$$

Corollary 103. *The Hopf surface X is a compact complex manifold that is not Kähler.*

Proof. Last time we saw that if X compact complex is Kähler then the even Betti numbers are nonzero. But in the computation above we saw that $b_4(X) = 1 \neq 0$. □

Recall now that the Kähler form on $\mathbb{C}^n$ was given

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

Definition 104. We say that a Hermitian metric on a Hermitian manifold (X, h) **osculates to order** k to the Euclidean metric if for each $x \in X$ there exists a neighborhood U of x with coordinates $z_1, \dots, z_n$ such that on U we can write

$$h_{ij} = \delta_{ij} + \psi_{ij}$$

where ψ_{ij} vanishes to order $\geq k$ at x.

We also tend to write $\psi_{ij} = \mathcal{O}(|z|^k)$, thinking of the expression above as a Taylor expansion (in z and $\bar{z}$ as h is only smooth).

The following result is a useful local criterion for checking whether a metric is Kähler.

Proposition 105. *A Hermitian manifold (X, h) is Kähler if and only if h osculates to order 2 to the Euclidean metric on $\mathbb{C}^n$.*

We first need a lemma.

Lemma 106. *A Hermitian metric h is Kähler if and only if*

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$$

for all $j, k, l = 1, \ldots, n$.

Proof. The metric h is Kähler if and only if $d\omega = 0$. We have

$$\begin{aligned} d\omega &= d\left(\frac{i}{2}\sum_{jk} h_{jk} dz^j \wedge d\bar{z}^k\right) = 0 \\ &= \frac{i}{2}\sum_{jkl} \frac{\partial h_{jk}}{\partial z_l} dz^l \wedge dz^j \wedge d\bar{z}^k + \frac{i}{2}\sum_{jkl} \frac{\partial h_{jk}}{\partial \bar{z}_l} dz^j \wedge d\bar{z}^k \wedge d\bar{z}^l \end{aligned}$$

Hence h is Kähler if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \quad \text{and} \quad \frac{\partial h_{jk}}{\partial \bar{z}_l} = \frac{\partial h_{lk}}{\partial \bar{z}_j}.$$

But notice that the second equality is just a conjugation of the first. □

We now prove the proposition.

Proof. Suppose we have coordinates in which $h_{ij} = \delta_{ij} + \psi_{ij}$. Then $d\omega$ has coefficients $\partial\psi_{ij}/\partial z_k = 0$ at x since ψ_{ij} vanishes to order ≥ 2 whence $d\omega = 0$ and h is Kähler.

Conversely assume $d\omega = 0$. Fix an $x \in X$ and local coordinates $z_1, \ldots, z_n$ centered at x. Perform a linear change of coordinates such that $h_{jk}(0) = \delta_{jk}$. Hence we can locally write

$$h_{jk} = \delta_{jk} + a_{jk} + \bar{a}_{kj} + \mathcal{O}(|z|^2)$$

where a_{jk} is a linear form in the z_i and $\bar{a}_{kj}$ is a linear form in the $\bar{z}_i$. By the lemma above we know that

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}.$$

Applying this equation to the expression for h above we have

$$\frac{\partial a_{jk}}{\partial z_l} = \frac{\partial a_{lk}}{\partial z_j}$$

from which it follows that we have a quadratic form $q_k(z)$ for each $k = 1, \ldots, n$ such that $a_{jk} = \partial q_k/\partial z_j$ and $q_k(0) = 0$. Now make a change of coordinates

$$w_j = z_j + q_j(z).$$

This is indeed a coordinate change as the Jacobian is just the identity at the origin. Now if we compute ω in these new coordinates we find that, up to terms $\mathcal{O}(|z|^2)$

that we may ignore for our purposes,

$$\omega = \frac{i}{2}\sum_{jk=1}^{n} h_{jk} dz^j \wedge d\bar{z}^k \simeq \frac{i}{2}\sum_{jk}^{n}(a_{jk}+\bar{a}_{kj})dz^j \wedge d\bar{z}^k$$

$$+\frac{i}{2}dz^j \wedge d\bar{z}^k + \frac{i}{2}\sum_{jk}^{n} dz^j \wedge \bar{a}_{kj} d\bar{z}^j + \frac{i}{2}\sum_{jk}^{n} a_{kj} dz^k \wedge d\bar{z}^j$$

$$\simeq \frac{i}{2}\sum_{j=1}^{n} dw^j \wedge d\bar{w}^j,$$

as desired. □

This criterion will come in handy later when we discuss the Hodge decomposition. For now we will take a break and discuss some abstract theory.

12.2. **Cohomology of sheaves.** Consider the following motivation/example, known as the Mittag-Leffler problem.

Let X be a Riemann surface and consider a function $f : X \to \mathbb{C} \cup \{\infty\}$ such that locally $f = g/h$ for $h \not\equiv 0$ with h, g holomorphic. Indeed, for such a function, for any x we can find coordinates z such that

$$f(z) = \sum_{k \geq -m} a_k z^k.$$

We denote the "polar part"

$$\mathcal{P}_x(f) = \sum_{-m \geq k \geq -1} a_k z^k$$

the negative powers.

Mittag-Leffler problem: given a discrete set of points $x_1, x_2, \ldots$ on X and prescribed polar parts $\mathcal{P}_1, \mathcal{P}_2, \ldots$ is there a function f holomorphic away from the x_i and such that $\mathcal{P}_{x_i}(f) = \mathcal{P}_i$?

We will approach this problem by one simple remark after another until we reach the notion of cohomology. Fix neighborhoods $U_i \ni x_i$ such that $x_j \notin U_i$ for $j \neq i$. Moreover we write $U_0 = X \setminus \{x_1, x_2, \ldots\}$ and $\mathcal{P}_0 = 0$. On the double overlaps $U_i \cap U_j$ we define $g_{ij} = \mathcal{P}_i - \mathcal{P}_j$. As we are away from the problem points the g_{ij} are holomorphic. Notice that moreover on $U_i \cap U_j \cap U_k$ we have that

$$g_{ij} + g_{jk} + g_{ki} = 0.$$

Now suppose there exists a function f satisfying our conditions. Then $f - \mathcal{P}_i \in \mathcal{O}_X(U_i)$. But then on the overlap $U_i \cap U_j$ we obtain a holomorphic function

$$g_{ij} = (f - \mathcal{P}_j) - (f - \mathcal{P}_i).$$

We conclude that we have a collection g of holomorphic functions on the cover $\mathcal{U}$ and that the Mittag-Leffler problem has a solution if and only if $[g] = 0$ in $H^1(\mathcal{U}, \mathcal{O}_X)$, the first Čech cohomology group of $\mathcal{O}_X$ with respect to the open cover $\mathcal{U}$:

$$H^1(\mathcal{U}, \mathcal{O}_X) = \frac{\{g \mid g_{ij} + g_{jk} + g_{ki} = 0\}}{\{g \mid g_{ij} = f_j - f_i\}}.$$

This is useful because we will have general criteria for vanishing of higher cohomologies.

13. February 9, 2018

Recall last time we argued that the Mittag-Leffler problem had a solution if and only if a certain cocycle g on $U_i \cap U_j$ is a cobundary in $H^1(\mathcal{U}, \mathcal{O}_X)$.

Let's think about this formalism in a case we're a little bit more familiar with. Let ρ_i be a bump function on $U_i \ni x_i$ (where recall U_j does not contain x_i for $i \neq j$). Assume that $\rho_i \equiv 1$ identically in a small neighborhood of x_i in U_i. Consider

$$\omega = \sum_{i=0}^{\infty} \bar{\partial}(\rho_i \mathcal{P}_i),$$

which is a smooth form of type $(0,1)$ that is identically zero in a neighborhood of x_i and such that $\bar{\partial}\omega = 0$. This gives us a class $[\omega] \in H^{0,1}X$, which is trivial in cohomology if and only if $\omega = \bar{\partial}\phi$ for $\phi \in C^\infty(X)$, i.e. if and only if

$$\bar{\partial}(\phi - \sum \rho_i \mathcal{P}_i) = 0.$$

This condition is equivalent to the holomorphicity of $\phi - \sum \rho_i \mathcal{P}_i$, which in turn shows that ϕ is a solution to the Mittag-Leffler problem. Recall that we have already seen that $H^{0,1}\mathbb{C} = 0$ by the $\bar{\partial}$-Poincaré lemma. Hence the Mittag-Leffler problem has a solution on $\mathbb{C}$, which is already a nontrivial result.

Of course, we expect that these two different cohomological conditions are equivalent. Indeed, we will see that $H^1(\mathcal{U}, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = H^{0,1}X$. But let us now discuss sheaves and cohomology more carefully.

13.1. Sheaves.

We now introduce the language of sheaves. Sheaves are useful as they allow us to deal with and relate geometric objects defined on all open subsets of a manifold at the same time.

Definition 107. Let X be a topological space. A **sheaf** $\mathcal{F}$ of abelian groups (sets, rings, modules, etc.) is an assignment to each open set $U \subset X$ an abelian group $\mathcal{F}(U)$ of "sections on U" together with maps of abelian groups for each pair of opens $V \subset U \subset X$

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that if $W \subset V \subset U$ are opens then $\rho_W^U = \rho_W^V \circ \rho_V^U$ and such that if $s_i \in \mathcal{F}(U_i)$ for some $\{U_i\}$ an open cover of U such that $s_i|_{U_i \cap U_h} = s_j|_{U_i \cap U_j}$ then there exists a unique $s \in \mathcal{F}(U)$ restricting to s_i on each U_i.

Remark 108. Notice that we may write $s|_V := \rho_V^U s$. Moreover we will use the notation $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$.

Example 109.
1. For every geometric structure $(X, \mathcal{O}_X)$ we obtain a sheaf $\mathcal{O}_X$ of rings on X.
2. The locally constant sheaf with values in an abelian group A sends $U \mapsto \mathcal{F}(U) = \{f : U \to A\}$ where A is given the discrete toplogy and we take continuous maps. We will denote this sheaf as $\underline{A} = \underline{A}_X$. For instance $\Gamma(X, \underline{C}_X) = \mathbb{C}^{|\pi_0 X|}$.
3. If $\pi : V \to X$ is a vector bundle then we have an associated sheaf of sections of V denoted $\mathcal{V}$ where

$$\mathcal{V}(U) = \{s : U \to V \mid \pi \circ s\}$$

where the regularity of the map is determined by the regularity of the vector bundle. In fact this is a sheaf of $\mathcal{O}_X$-modules.

For instance, if $V = T_{\mathbb{R}}X$ then $\mathcal{V}(U)$ is the space of smooth vector fields on U. If $V = \Lambda^p T^*_{\mathbb{R}}X$ then $\mathcal{V}(U)$ is the space of smooth p-forms on U. The standard notation for this latter sheaf is $\mathcal{A}^p_X = \mathcal{A}^p$. Similarly the sheaf of (p,q)-forms is written $\mathcal{A}^{p,q}_X = \mathcal{A}^{p,q}$.

Notice that given a complex manifold X we obtain not only $\mathcal{O}_X$, the sheaf of holomorphic functions, but also $\mathcal{O}_X^\times$ the sheaf of nowhere-vanishing (invertible) holomorphic functions, and $\Omega^p_X = \mathcal{A}^{p,0}_X$ the sheaf of holomorphic p-forms.

Definition 110. Let $\mathcal{F}$ be a sheaf on X and $x \in X$ be a point. Then we define the **stalk** of $\mathcal{F}$ at x to be

$$\mathcal{F}_x := \operatorname*{colim}_{U \ni x} \mathcal{F}(U).$$

For example, $\mathcal{O}_{X,x}$ is just the ring of germs of holomorphic functions on a complex manifold (which we know is isomorphic to $\mathbb{C}\{z_1, \dots, z_n\}$). Notice that the local information here is the same at each point!

Definition 111. A **map of sheaves** $f : \mathcal{F} \to \mathcal{G}$ is a collection of maps of abelian groups $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each $U \subset X$ such that for each $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

commutes. If all f_U are injective we say that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$.

One can check that the kernel of the map of sheaves is a sheaf, which we denote $\ker f$. The same is not true, however, of the image of a map f. The image is only what is called a presheaf, so one has to "sheafify". Concretely one finds that the resulting object has

$$\operatorname{im} f(U) = \{s \in \mathcal{G}(U) \mid \exists \{U_i\} \supset U s|_{U_i} \in \operatorname{im} f_{U_i}\}.$$

Definition 112. We say that $f : \mathcal{F} \to \mathcal{G}$ is injective if $\ker f = 0$ and surjective if $\operatorname{im} f = \mathcal{G}$. We say that a sequence of maps of sheaves

$$\mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2 \to \cdots$$

is a complex if $d^i \circ d^{i-1} = 0$. We say that the sequence is exact if $\operatorname{im} d^{i-1} = \ker d^i$ for each i.

One has to be careful – exactness of a complex of sheaves is only equivalent to exactness of the corresponding complex of stalks at each point. One cannot say anything in general about the corresponding sequence of sections over some open U.

Exercise 113. We leave the following as a homework exercise: a complex $(\mathcal{F}^\bullet, d^\bullet)$ is exact if and only if the corresponding sequence of stalks is exact for all $x \in X$. Find a counterexample for the corresponding statement for sections over an open.

The following example is very important.

Example 114. Consider the (short exact) **exponential sequence** of sheaves on a complex manifold X:

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{\exp(2\pi i -)} \mathcal{O}_X^\times \to 1$$

Here the exponential map, on each U open, is given $f \mapsto \exp(2\pi i f)$. Let us check that the sequence is exact at $\mathcal{O}_X$: suppose $e^{2\pi i f} = 1$, then f is simply a (locally) constant integer. For exactness at $\mathcal{O}_X^\times$ we have to check that the last map is surjective. Here is where we need to reduce to stalks (it is certainly not true that we can solve $\exp(2\pi i f = g)$ of sections near 0), to locally take a logarithm.

14. February 12, 2018

Recall last time we were discussing the exponential short exact sequence. The subtle point here was that the surjectivity of the map $\mathcal{O}_X \to \mathcal{O}_X^\times$ followed from the *local* existence of logarithms (i.e. at the level of stalks).

Lemma 115. *Given a short exact sequence*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$$

of sheaves on X, we obtain an exact sequence of abelian groups

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

for each $U \subset X$ open.

Proof. Left as an exercise. □

In particular, if we are given a short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

then we obtain an exact sequence

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3).$$

The failure of exactness at the right is corrected by the presence of higher sheaf cohomology groups. Indeed, we obtain a long exact sequence in sheaf cohomology.

14.1. Constructing sheaf cohomology.

Definition 116. A sheaf $\mathcal{F}$ on X is called **flabby** or **flasque** if the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for each $U \subset X$ open. It follows that all restriction maps are surjective.

Lemma 117. *If we have a short exact sequence sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

such that $\mathcal{F}_1$ is flabby then we also obtain a short exact sequence on global sections

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0.$$

Proof. We need to show that $\Gamma(X, \mathcal{F}_2)$ surjects onto $\Gamma(X, \mathcal{F}_3)$. Choose a global section $t \in \Gamma(X, \mathcal{F}_3)$. Consider the set

$$\mathcal{A} = \{(U, s) \mid s \in \mathcal{F}_2(U), s \mapsto t|_U\}.$$

Notice that $\mathcal{F}_{2,x} \twoheadrightarrow \mathcal{F}_{3,x}$ for all x. It follows that $\mathcal{A}$ is nonempty: we can always choose a germ over a small enough open set. There is an obvious partial order on this set where

$$(U_1, s_1) \leq (U_2, s_2) \iff U_1 \subset U_2, s_2|_{U_1} = s_1.$$

Now we use the sheaf property to apply Zorn's lemma: since $\mathcal{F}_2$ is a sheaf, for every chain $(U_i, s_i)_{i \in I}$ in $\mathcal{A}$ there exists an upper bound given by $U = \cup_{i \in I} U_i$ and s such that $s|_{U_i} = s_i$ given uniquely by the sheaf property. Hence $\mathcal{A}$ has a maximal element, call it $(\bar{U}, \bar{s})$. It remains to show that $\bar{U} = X$.

Suppose $\bar{U} \neq X$, i.e. there exists a point $x \in X \setminus \bar{U}$. There exists $(U, s) \in \mathcal{A}$ with $x \in U$ by the surjectivity on stalks as before. Define $V = U \cap \bar{U}$. If V is empty we are done, so suppose otherwise. Both $\bar{s}|_V$ and $s|_V$ map to $t|_V$ by construction. Hence $\bar{s}|_V - s|_V$ is in the kernel of the map $\mathcal{F}_2(V) \to \mathcal{F}_3(V)$ whence by exactness we obtain a section $w \in \mathcal{F}_1(V)$ such that $f_V(w) = (\bar{s} - s)|_V$ (recall injectivity goes through on sections, not just stalks). Now, using that $\mathcal{F}_1$ is flabby there exists a section $w' \in \mathcal{F}_1(U)$ such that $w'|_V = w$. Notice that $\bar{s}$ and $f_U(w') + s$ both map to t and they agree on V. The sheaf property of $\mathcal{F}_2$ yields a gluing $s''|_{U \cup \bar{U}}$ such that $s'' \mapsto t$. Since $(\bar{U}, \bar{s}) \leq (U \cup \bar{U}, s'')$ we contradict the maximality of $(\bar{U}, \bar{s})$. $\square$

Definition 118. A (cohomologically indexed) **resolution** of a sheaf a $\mathcal{F}$ is a complex of sheaves

$$\mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

together with a map $\mathcal{F} \hookrightarrow \mathcal{F}_0$ such that the sequence

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

is exact.

Proposition 119 (Godement resolution)**.** *Any sheaf $\mathcal{F}$ on a topological space X admits a resolution by flabby sheaves.*

Proof. It is enough to show that we can embed any $\mathcal{F}$ into a flabby sheaf because then we can embed $\mathcal{F} \to \mathcal{F}^0$, take the cokernel $\mathcal{F}^0/\mathcal{F}$ and then embed that into a flabby sheaf, and so on.

Indeed there is a canonical map $\mathcal{F} \hookrightarrow \mathsf{ds}(\mathcal{F})$ into the sheaf of discontinuous sections,

$$\mathsf{ds}(\mathcal{F}) : U \mapsto \{s : U \to \coprod_{x \in U} \mathcal{F}_x, \pi \circ s = \mathrm{id}_U\}.$$

The map into this sheaf sends a section s over U to $\coprod_{x \in U} s_x$. This sheaf is clearly flabby: there is no regularity condition (hence "discontinuous") on the sections so we can always extend sections arbitrarily. $\square$

With this canonical flabby resolution of any sheaf we can now define the cohomology of sheaves.

Definition 120. Let $\mathcal{F}$ be a sheaf on a topological space X. Let $\mathcal{F} \to \mathcal{F}^\bullet$ be the Godement resolution. Then the i**th cohomology group of** $\mathcal{F}$ is the ith cohomology of the complex

$$0 \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2) \to \cdots$$

which we denote as $H^i(X, \mathcal{F})$.

Corollary 121. *We have $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.*

Proof. Clear from the definition. $\square$

Proposition 122. *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence then we obtain a long exact sequence on sheaf cohomology groups.*

Proof. This follows from general homological algebra and left-exactness. $\square$

Flabby sheaves are quite simple cohomologically: they are acyclic.

Lemma 123. *If $\mathcal{F}$ is flabby then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Left as an exercise: look at the cokernels of the maps in a flabby resolution, and use the lemma below. $\square$

Lemma 124. *Given a short exact sequence of sheaves, if the first two sheaves are flabby then the third is flabby as well.*

15. February 14, 2018

Recall last time we considered, for a sheaf $\mathcal{F}$ on X, flabby resolutions $\mathcal{F} \to \mathcal{F}^\bullet$. In particular we defined sheaf cohomology

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{F}^\bullet)$$

as the cohomology of the complex obtained by applying the global sections functor to the resolution $\mathcal{F}^\bullet$. It turns out, in fact, that we don't have to use flabby resolutions—we may as well use a resolution by acyclic sheaves (recall that flabby sheaves happen to be acyclic).

Let's return to the exponential sequence:

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

Recall that any short exact sequence of sheaves yields a long exact sequence on sheaf cohomologies. It is a fact that sheaf cohomology simply computes the singular cohomology of X:

$$H^i(X, \underline{\mathbb{Z}}_X) \cong H^i(X; \mathbb{Z}).$$

Hence integral singular cohomology appears in this long exact sequence. Since we know how to compute these groups for some nice spaces, let's look at some examples.

Example 125. Suppose X is a simply connected subset of $\mathbb{C}^n$. Then $H^1(X; \mathbb{Z}) = 0$. Hence the long exact sequence yields a short exact sequence

$$0 \to H^0(X; \mathbb{Z}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times) \to 0$$

In other words we recover the fact that we can define the logarithm globally on X.

Example 126. If X is compact connected then $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X, \mathcal{O}_X^\times) = \mathbb{C}^\times$. Thus we obtain

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1$$

and an exact sequence

$$0 \to H^1(X, \underline{\mathbb{Z}}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \to \cdots$$

This sequence is extremely important as the left-hand side encodes the Picard variety and the right-hand side encodes data of line bundles up to isomorphism, the Picard group.

15.1. The Dolbeault theorem.

We will now relate the sheaf cohomology of the sheaves of differential forms on a complex manifold X to the Dolbeault cohomologies that we defined earlier. Recall that these were defined to be

$$H^{p,q}X = \frac{\ker(\bar{\partial} : A^{p,q}X \to A^{p,q+1})}{\operatorname{im}(\bar{\partial} : A^{p,q-1}X \to A^{p,q}X)}.$$

We also have the bundle Ω_X^p of holomorphic p-forms (and its sheaf of sections).

Theorem 127 (Dolbeault theorem)**.** *Let X be a complex manifold. Then, for every p, q*

$$H^{p,q}X \cong H^q(X, \Omega_X^p).$$

Proof. Recall that we have the sheaf of smooth (p,q)-forms $\mathcal{A}_X$ on X. We claim that there is a resolution of the sheaf Ω_X^p,

$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots \to \mathcal{A}_X^{p,n} \to 0$$

Exactness is clear at $\mathcal{A}_X^{p,0}$. The exactness at each other term is exactly the $\bar\partial$-Poincaré lemma. Indeed, checking exactness is equivalent to checking exactness on stalks, but the lemma tells us that locally every $\bar\partial$-closed form is $\bar\partial$-exact. We claim that

$$H^i(X, \mathcal{A}_X^{p,q}) = 0$$

for all $i > 0$. This follows from the fact that $\mathcal{A}_X^{p,q}$ admit partitions of unity. We will prove this next time. Hence we obtain an acyclic resolution of Ω_X^p whence we are done.

However, we have not shown that acyclic resolutions compute cohomology, so let's do it explicitly. We define K_q to fit into the exact sequence of sheaves (for $q > 0$)

$$0 \to K_q \to \mathcal{A}_X^{p,q} \to K_{q+1} \to 0$$

(splicing our resolution) whence we obtain a long exact sequence on cohomology. This implies, by acyclicity of $\mathcal{A}_X^{p,q}$,

$$H^{q-1}(X, K_1) \cong \cdots \cong H^q(X, K_{q-1}).$$

We define $K_0 = \Omega_X^p$. Now it is enough to show that

$$H^1(X, K_q) \cong H^{p,q+1}X.$$

We have an exact sequence

$$0 \to K_q \to \mathcal{A}_X^{p,q} \xrightarrow{\bar\partial} K_{q+1} \to 0$$

Writing out the long exact sequence on sheaf cohomologies we obtain

$$H^0(X, \mathcal{A}_X^{p,q}) \to H^0(X, K_{q+1}) \to H^1(X, K_q) \to 0$$

Because we have an exact sequence

$$0 \to H^0(X, K_{q+1}) \to A^{p,q+1}X \xrightarrow{\bar\partial} A^{p,q+2}X$$

we find that

$$H^1(X, K_q) \cong \frac{H^0(X, K_{q+1})}{\operatorname{im} \bar\partial|_{A^{p,q}X}} = H^{p,q}X.$$

□

15.2. **Čech cohomology.** So far we don't have very good tools for computing sheaf cohomology groups. Čech cohomology will be one such tool.

Definition 128. Let X be a topological space and $\mathcal{F}$ a sheaf on X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Define the pth Cech group

$$C^p(\mathcal{U}, F) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_i \cap \cdots \cap U_{i_p}).$$

We define a differential

$$d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

sending

$$g \mapsto h, \quad h_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k g_{i_0 \cdots \hat{i}_k \cdots i_{p+1}} |_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}.$$

We leave it as an exercise to check that $d^2 = 0$. Thus we obtain a complex of abelian groups

$$(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), d^\bullet),$$

The Čech cochain complex of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$. Finally we define the Čech cohomology

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F})).$$

Next time we will see how the Čech cohomology computes sheaf cohomology under certain assumptions on the cover $\mathcal{U}$. Just a reminder: there will be no class next Monday.

16. February 16, 2018

16.1. **Čech and sheaf cohomology.** Last time we had fixed a sheaf $\mathcal{F}$ on a topological space X and an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and given this data, defined the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$. Let's compute some concrete examples to get a feel for these groups.

Example 129. Notice that $\check{H}^0(\mathcal{U},\mathcal{F}) = H^0(X,\mathcal{F}) = \Gamma(X,F)$. To see this, notice that $\check{C}^0(\mathcal{U},\mathcal{F}) \to \check{C}^1(\mathcal{U},\mathcal{F})$, takes $(s_i) \mapsto s_j - s_i|_{U_i\cap U_j}$. The kernel of this map is precisely the collections of sections s_i on U_i that agree on their overlaps. By the sheaf property this is exactly the space of global sections.

Example 130. Consider the holomorphic line bundles on X a complex manifold trivialized over a cover $\{U_i\}_{i\in I}$ with transition functions $g_{ij} \in \mathcal{O}_X^\times(U_i\cap U_j)$ such that $g_{ij}g_{jk}g_{ki} = 1$. Notice that $g = (g_{ij}) \in \check{C}^1(\mathcal{U},\mathcal{O}_X^\times)$ is a cocycle. Hence g defines a class $[g] \in \check{H}^1(\mathcal{U},\mathcal{O}_X^\times)$. This gives us a map from the set of holomorphic line bundles on X (trivialized over $\mathcal{U}$ to the $\check{H}^1(\mathcal{U},\mathcal{O}_X^\times)$ (in fact a group homomorphism). We claim that if we pass to the isomorphism classes of the line bundles then this map becomes an isomorphism.

To check injectivity we notice that g is a coboundary if there exist $s_i \in \mathcal{O}_X^\times(U_i)$ $g_{ij} = s_j/s_i$. But this condition precisely yields a nonvanishing global section of L! This implies that L is trivial. Surjectivity of the map is clear, as any such Čech data yields a line bundle.

If we want to forget about the choice of cover, we can take a limit over open covers and refinements — this yields the Picard group of X on the left (the group of all holomorphic line bundles on X up to isomorphism under tensor product) and the sheaf cohomology $H^1(\mathcal{U},\mathcal{O}_X^\times)$. We will justify this later.

The following result follows fairly straightforwardly from a spectral sequence argument.

Theorem 131 (Cartan's lemma, Leray's theorem)**.** *Let X be a space and $\mathcal{F}$ be a sheaf on X. Assume that $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X such that $\mathcal{U}$ is acyclic for $\mathcal{F}$, i.e.*

$$H^i(U_{i_1}\cap\cdots\cap U_{i_p},\mathcal{F}) = 0, \quad \textit{for all } i > 0,\ \textit{for all } p \geq 1,\ \textit{for all } i_j.$$

Then

$$\check{H}^i(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^i(X,\mathcal{F})$$

is an isomorphism.

In fact it is true that $H^i(X,\mathcal{F}) \cong \operatorname{colim}_{\mathcal{U}} \check{H}^i(\mathcal{U},\mathcal{F})$ in general.

Example 132. Let $X = \mathbb{P}^1$. We'd like to understand the cohomology of the trivial line bundle $\mathcal{O}_{\mathbb{P}^1}$. We already know that $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}$. What about H^1? Take U_0, U_1 the standard open cover of $\mathbb{P}^1$. We have that $U_0 \cong U_1 \cong \mathbb{C}$ and $U_0\cap U_1 = \mathbb{C}^\times$. To apply the theorem above we need to make sure that

$$H^i(\mathbb{C},\mathcal{O}_{\mathbb{C}}) = H^i(\mathbb{C}^\times,\mathcal{O}_{\mathbb{C}^\times}) = 0$$

for $i > 0$. To see this we apply the Dolbeault theorem, which tells us that

$$H^i(X,\mathcal{O}_X) = H^{0,i}(X).$$

The $\bar{\partial}$-Poincaré lemma (and the version for $\mathbb{C}^\times$ in the homework) thus shows that we have an acyclic cover. Hence we conclude that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}).$$

The Čech complex in our case is very simple.

$$0 \to \mathcal{O}_{\mathbb{P}^1}(U_0) \oplus \mathcal{O}_{\mathbb{P}^1}(U_1) \to \mathcal{O}_{\mathbb{P}^1}(U_1 \cap U_2) \to 0$$

The map is given $(f, g) \mapsto g - f$. We claim that this map is surjective, whence

$$\check{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0.$$

The claim follows from simple complex analysis, the holomorphic functions on $\mathbb{C}^\times$ are precisely the Laurent polynomials in z and $1/z$. Or something like that.

Moreover one can show that, more painfully,

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0, \quad i > 0.$$

Similarly one might want to look at $\mathcal{O}_{\mathbb{P}^n}(m)$. This computation can be found, for example, in Hartshorne.

We still have not shown that

$$H^i(X, \mathcal{A}_X^{p,q}) = 0, \quad i > 0,$$

that we used last time, but we will show this next time.

16.2. **The Cousin problem.** Let's look at a practical example. Let $X \subset \mathbb{C}^n$ be an analytic hypersurface, i.e. a subset given locally as the zero set of one holomorphic function. There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $\mathbb{C}^n$ with functions $f_i \in \mathcal{O}_X(U_i)$ such that $X \cap U_i = Z(f_i)$. We claim that $X = Z(f)$ for $f \in \mathcal{O}(\mathbb{C}^n)$, i.e. X can be written globally as the zero set of one function.

Let's first make a few general remarks about $\mathbb{C}^n$ with a cover $\mathcal{U}$. We can always refine the cover such that each U_i is an open polybox,

$$U_i = \{z \in \mathbb{C}^n \mid |x_j - a_j| < r_j, |y_j - b_j| < s_j\}.$$

It is easy to see that iterated intersections are also polyboxes. Hence all the iterated intersections are contractible. Now

$$H^q(V, \Omega_V^p) \cong H^{p,q}V = 0$$

where V is any such iterated intersection, for each $q > 0$ by the Dolbeault theorem and the $\bar{\partial}$-Poincaré lemma. Hence $\mathcal{U}$ is an acyclic cover for each $\Omega^p_{\mathbb{C}^n}$ for each p. By Leray's theorem and again the Dolbeault theorem, we conclude that

$$H^q(\mathbb{C}^n, \Omega^p_{\mathbb{C}^n}) \cong \check{H}^q(\mathcal{U}, \Omega^p_{\mathbb{C}^n}) \cong H^{p,q}\mathbb{C}^n = 0.$$

For $p = 0$ we find that $H^{0,q}\mathbb{C}^n \cong H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \cong \check{H}^q(\mathcal{U}, \mathcal{O}_{\mathbb{C}^n}) = 0$ for each $q > 0$. We will use this fact to solve the Cousin problem. This is a special case of the very general notion of Stein spaces.

17. February 21, 2018

17.1. **The Cousin problem, continued.** Recall the setup from last time. We want to exploit the vanishing results on cohomology that we obtained last time in order to show that our analytic hypersurface is globally a zero locus.

Recall on $U_j \cap U_k$ we have $f_j = g_{jk} f_k$ with $g_{jk} \in \mathcal{O}_{\mathbb{C}^n}^{\times}(U_j \cap U_k)$. Since $U_j \cap U_k$ is contractible we can solve

$$g_{jk} = \exp(2\pi i h_{jk})$$

for $h_{jk} \in \mathcal{O}_{\mathbb{C}^n}(U_i \cap U_j)$. This follows, for example, by the long exact sequence associated to the short exact exponential sequence (and using our knowledge of singular cohomology). On $U_j \cap U_k \cap U_l$ we know that

$$g_{jk} g_{kl} g_{lj} = 1,$$

which is equivalent to saying that

$$a_{jkl} := h_{jl} - h_{jk} - h_{kl} \in \mathbb{Z}$$

take values in $\mathbb{Z}$. One checks that $\{a_{jkl}\}$ forms a 2-cocycle by taking δ whence we obtain $a \in \check{H}^2(\mathcal{U}, \mathbb{Z})$. By the Leray theorem this group is just the singular cohomology $H^2(\mathbb{C}^n, \mathbb{Z}) = 0$. We conclude that a is a coboundary—there exist integers b_{ij} for all i, j such that

$$a_{jkl} = b_{kl} - b_{jl} + b_{jk}.$$

Now replace the h_{ij} by $h_{ij} + b_{ij}$ which of course does not change the g_{ij}. Now h yields satisfies the cocycle condition on $U_j \cap U_k \cap U_l$, i.e. $h \in \check{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{C}^n}) = 0$. This in turn means that we can write each $h_{jk} = h_k - h_j$ for $h_j \in \mathcal{O}_{\mathbb{C}^n}(U_j)$. Write, now:

$$f_j = \exp(2\pi i h_{jk}) f_k = \exp(2\pi i (h_k - h_j)) f_k$$

whence $f_j \exp(2\pi i h_j) = f_k \exp(2\pi i h_k)$ on $U_j \cap U_k$ for each j, k. Invoking the sheaf property we obtain a unique $f \in \mathcal{O}(\mathbb{C}^n)$ such that $f|_{U_j} = f_j \exp(2\pi i h_j)$ for all j. Since the exponential is nonvanishing we find that the zero locus of f is precisely the zero locus of f_j on each U_j. We conclude that $X = Z(f)$.

17.2. **A result I owe you.** Recall we haven't yet shown that $H^i(X, \mathcal{A}_X^{p,q}) = 0$ for all $i > 0$. We will need the fact that since X is a manifold it is paracompact, i.e. every open cover has a locally finite subcover.

Definition 133. Let $\mathcal{F}$ be a sheaf on a paracompact topological space X. We say that $\mathcal{F}$ is **fine** if for every locally finite open cover $\mathcal{U}$ of X there exist sheaf endomorphisms

$$\phi_i : \mathcal{F} \to \mathcal{F}$$

such that

(1) For each i there exist open sets V_i such that $X \setminus U_i \subset V_i$ and the map $\phi_{i,x}$ on stalks for each $x \in V_i$ is zero;
(2) as morphisms of sheaves $\sum_{i \in I} \phi_i = \mathrm{id}_{\mathcal{F}}$.

This notion is essentially encapsulating the notion of a partition of unity.

Lemma 134. *The sheaves $\mathcal{A}_X^{p,q}$ are fine.*

Proof. Fix a partition of unity $\{\rho_i\}$ subordinate to the given open cover and define ϕ_i to be given by multiplication by ρ_i. □

Remark 135. Notice that condition 1 in the definition of a fine sheaf is equivalent to having, for each section $s \in \mathcal{F}(X)$ we have that $\operatorname{supp} \phi_i(s) \subset U_i$. Condition 2 is equivalent to having that $s = \sum_{i \in I} \phi_i(s)$.

Remark 136. Let $s \in \mathcal{F}(U_i)$. Then $\phi_i(s) = 0$ near the boundary of U_i. Hence, by the sheaf property we obtain a global section by gluing with the zero section outside of U_i.

Proposition 137. *If $\mathcal{F}$ is a fine sheaf then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

The proof proceeds in 2 steps. First we show that $\mathcal{F}$ fine implies that $\mathcal{F}$ is soft. Then we show that soft sheaves are acyclic.

Definition 138. Let $\mathcal{F}$ be a sheaf on X. If $Z \subset X$ is a closed subset then we define $\mathcal{F}|_Z$ to be given by $(\mathcal{F}|_Z)_x = \mathcal{F}_x$ for all $x \in Z$. More precisely, to $U \subset Z$ open we assign the set of sections $s : U \to \coprod_{x \in U} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for each x and such that for all $x \in U$ s is locally the restriction of a section of $\mathcal{F}$ to Z.

Notice that this is the inverse image of $\mathcal{F}$ along the inclusion $Z \hookrightarrow X$.

Definition 139. If $\mathcal{F}$ is a sheaf on X paracompact then we call $\mathcal{F}$ **soft** if for each $Z \subset X$ closed, $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is surjective.

Example 140. For every $\mathcal{F}$ the sheaf $\mathsf{ds}(\mathcal{F})$ is soft. Hence the sheaves in the Godement resolution are both flabby and soft. In fact $\mathsf{ds}(\mathcal{F})$ is also fine.

Lemma 141. *If $\mathcal{F}$ is fine then $\mathcal{F}$ is soft.*

Proof. Let $Z \subset X$ be closed and take $t \in \Gamma(Z, \mathcal{F}|_Z)$. We wish to lift t to a global section on X. There exist $\{U_i\}$ such that $U_i \subset X$ open and $Z \subset \cup U_i$. We can shrink the U_i such that there exist $s_i \in \mathcal{F}(U_i)$ with

$$s_i|_{Z \cap U_i} = t|_{Z \cap U_i}.$$

Now, since X is paracompact we can assume that $\mathcal{U}$ is locally finite, and since $\mathcal{F}$ is fine, there exist $\phi_i : \mathcal{F} \to \mathcal{F}$ such that $\sum \phi_i = \mathrm{id}_{\mathcal{F}}$ and such that $\phi_i \equiv 0$ on some neighborhood V_i containing $X \setminus U_i$. The section $\phi_i(s_i)$ extends by zero to a global section. Define

$$s = \sum_{i \in I} \phi_i(s_i) \in \Gamma(X, \mathcal{F}).$$

For each $x \in Z$ we have that $(s_i)_x = t_x$. Finally

$$s_x = \sum_i \phi_i((s_i)_x) = \sum_i \phi_i(t_x) = t_x,$$

and we are done. □

18. February 23, 2018

18.1. **Soft sheaves, cont.**

Lemma 142. *If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves for $\mathcal{F}_1$ soft then we have induced short exact sequence on global sections.

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0$$

Proof. The proof is very similar to the flasque case. □

Remark 143. Let $i : Y \to X$ be a map. Then there is a functor i^{-1} taking sheaves on X to sheaves on Y called the inverse image. This functor is in particular exact as the stalks are preserved.

Lemma 144. *If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves where $\mathcal{F}_1, \mathcal{F}_2$ are soft then $\mathcal{F}_3$ is soft.

Proof. Fix $Z \subset X$ closed. We have a diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}_2) & \longrightarrow & \Gamma(X, \mathcal{F}_3) \\ \downarrow & & \downarrow \\ \Gamma(Z, \mathcal{F}_2|_Z) & \longrightarrow & \Gamma(Z, \mathcal{F}_3|_Z) \end{array}$$

that commutes. Notice that the left vertical arrow is a surjection and the upper horizontal arrow is as well. Using the fact that the inverse image is an exact functor we see that the right vertical arrow is a surjection, and we are done. □

Write this out more carefully

Proposition 145. *If $\mathcal{F}$ is soft then it is acyclic.*

Proof. Start with the Godement resolution (which is a soft resolution)

$$\mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

which is an exact sequence. We splice this resolution, looking at the cokernels $\mathcal{G}^i$ of each map, which are each soft. We thus have an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

whence $\mathcal{F}$ is acyclic. □

18.2. **Real harmonic theory.**

We now turn to the subject of finding canonical representatives in each cohomology class $[\omega] \in H^k_{\mathsf{dR}} X \cong H^k(X; \mathbb{R})$. We begin by reviewing some linear algebra.

Let V be a real vector space of dimension n equipped with an inner product, $g : V \times V \to \mathbb{R}$. Notice that g induces an isomorphism $\phi : V \to V^*$ sending $v \mapsto g(v, -)$. Under this isomorphism we induce a metric g^* on V^* defined by $g^*\phi(v_1), \phi(v_2)) = g(v_1, v_2)$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ for V. Then $\{\phi(e_1), \ldots, \phi(e_n)\}$ is an orthonormal basis for V^*. Of course we also induce inner products $g : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$ given by

$$g(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det(g(v_i, w_j))_{ij}$$

and extended by linearity. Notice that we have an orthonormal basis for $\Lambda^k V$ given by $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ for all $i_1 < \cdots < i_k$.

Suppose we fix an orientation of V in which $e_1, \ldots, e_n$ is a positive basis with associated fundamental element $e_1 \wedge \cdots e_n \in \Lambda^n V \cong \mathbb{R}$.

Definition 146. The **Hodge star operator** is the unique linear endomorphism of $\Lambda^\bullet V$ such that $* : \Lambda^k V \to \Lambda^{n-k} V$ satisfies

$$\alpha \wedge *\beta = g(\alpha, \beta) e_1 \wedge \cdots \wedge e_n$$

for all $\alpha, \beta \in \Lambda^k V$.

On the chosen orthonormal basis we can write a formula for the Hodge star. Notice that For each $\sigma \in S_n$ we have

$$e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} = \operatorname{sgn}(\sigma) e_1 \wedge \cdots \wedge e_n.$$

Then

$$*(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}) = \operatorname{sgn}(\sigma) e_{\sigma(k+1)} \wedge \cdots \wedge e_{\sigma(n)}.$$

It is immediate that $*$ is an isomorphism in each degree, and in particular an isometry: $g(*\alpha, *\beta) = g(\alpha, \beta)$.

Lemma 147. *For each $\alpha \in \Lambda^k V$ we have that*

$$**\alpha = (-1)^{k(n-k)} \alpha.$$

In other words, $*^{-1} = (-1)^{k(n-k)} *$.

Proof. It is enough, since $*$ is an isomorphism, to wedge against every $\beta \in \Lambda^k V$:

$$\begin{aligned} **\alpha \wedge *\beta &= (-1)^{k(n-k)} * \beta \wedge ** \alpha \\ &= (-1)^{k(n-k)} g(*\beta, *\alpha) e_1 \wedge \cdots \wedge e_n \\ &= (-1)^{k(n-k)} g(\alpha, \beta) e_1 \wedge \cdots \wedge e_n \\ &= (-1)^{k(n-k)} \alpha \wedge *\beta \end{aligned}$$

as desired. □

Remark 148. One might call $* : \Lambda^k V \xrightarrow{\sim} \Lambda^{n-k} V$ the abstract Poincaré duality map.

Let us now globalize these linear algebraic constructions. Let X be a real manifold of dimension n equipped with a Riemannian metric g. We obtain metrics on the wedge products $\Lambda^k T^* X$ and in particular we obtain Hodge star operators at each point:

$$*_x : \Lambda^k T^*_x X \to \Lambda^{n-k} T^*_x X.$$

Globally this gives us $* : A^k X \to A^{n-k} X$ given by

$$\alpha \wedge *\beta = g(\alpha, \beta) \operatorname{vol}(g).$$

Define now the L^2 inner product on $\mathcal{A}^k X$ where

$$(\alpha, \beta)_X := \int_X \alpha \wedge *\beta = \int_X g(\alpha, \beta) \operatorname{vol}(g),$$

where we should make some sort of compactly supported assumption.

Proposition 149. *The operator* $d : A^{k-1}X \to A^kX$ *has a (formal) adjoint that we denote* $d^* : A^kX \to A^{k-1}X$ *with respect to the* L^2 *inner product, given by*

$$d^* = -(-1)^{n(k+1)} * d * .$$

Proof. We check that $(d\alpha, \beta)_X = (\alpha, d^*\beta)_X$ for each $\alpha \in A^{k-1}X, \beta \in A^kX$. We remark first that by Stokes' theorem

$$0 = \int_X d(\alpha \wedge *\beta) = \int_X d\alpha \wedge *\beta + (-1)^{k-1} \int_X \alpha \wedge d * \beta.$$

Hence

$$(d\alpha, \beta)_X = (-1)^k \int_X \alpha \wedge d * \beta = \int_X \alpha \wedge *\gamma = (\alpha, \gamma)_X$$

where $\gamma = (-1)^k *^{-1} d* = (-1)^k(-1)^{(n-k+1)(k-1)} * d * \beta$. But evaluating the signs one finds that $\gamma = d^*\beta$. □

Remark 150. Notice that d is a linear differential operator on X. There is always a unique formal adjoint with respect to the inner product on X, so this is just a special case of a general fact.

19. February 26, 2018

19.1. **Hodge theory for smooth manifolds.** Recall we are assuming our manifolds are compact.

Definition 151. The **Laplacian** on a compact Riemannian manifold X is the operator

$$\Delta = d \circ d^* + d^* \circ d : A^k X \to A^k X$$

for each $0 \leq k \leq n$. We say that a form ω is **harmonic** if

$$\Delta\omega = 0.$$

Example 152. The following is in the third homework. On $\mathbb{R}^n$, with respect to the Euclidean metric, we have

$$\Delta f = -\sum_i \frac{\partial^2 f}{\partial x_i^2}.$$

There are a few properties of the Laplacian to keep in mind. The Laplacian Δ is

(1) a second order linear differential operator
(2) in fact an elliptic operator
(3) self-adjoint with respect to the L^2 inner product.

This last point we can check easily by the formal adjointness of d and d^*

$$\begin{aligned}(\Delta\alpha, \beta)_X &= (dd^*\alpha + d^*d\alpha, \beta)_X = (dd^*\alpha, \beta)_X + (d^*d\alpha, \beta)_X \\ &= (d^*\alpha, d^*\beta)_X + (d\alpha, d\beta)_X \\ &= (\alpha, \Delta\beta)_X\end{aligned}$$

We will use a big theorem from the theory of PDEs which is more-or-less immediate from elliptic regularity.

Theorem 153. *Define $\mathcal{H}^k(X) = \ker\Delta \subset A^k X$ to be the space of harmonic k-forms. Then $\mathcal{H}^k X$ is a finite-dimensional vector space. Moreover, we have a splitting*

$$A^k(X) = \ker\Delta \oplus \operatorname{im}\Delta.$$

Remark 154. Notice that ω is harmonic if and only if $d\omega = d^*\omega = 0$. This follows immediately from computing:

$$\begin{aligned}0 = (\Delta\omega, \omega) &= (dd^*\omega, \omega)_X + (d^*d\omega, \omega)_X \\ &= ||d\omega||_X + ||d^*\omega||_X \geq 0.\end{aligned}$$

Hence each harmonic form ω yields a de Rham cohomology class $[\omega]$.

Theorem 155 (Hodge theorem for compact Riemannian manifolds)**.** *Let (X, g) be a compact oriented Riemannian manifold. Then for each k the mapping ϕ : $\mathcal{H}^k X \to H^k(X, \mathbb{R})$ defined in the remark above is an isomorphism. In other words, every de Rham cohomology class admits a unique harmonic representative.*

Proof. We start by showing that ϕ is injective. Suppose $[\omega] = 0$ i.e. $\omega = d\eta$ for some $\eta \in A^{k-1}X$. Now

$$||\omega||_X^2 = (\omega, d\eta)_X = (d^*\omega, \eta)_X = 0$$

whence $\omega = 0$.

Showing that ϕ is surjective is the nontrivial content of this theorem. Let $[\nu] \in H^k_{\mathsf{dR}}X$. Using the PDE theorem from above we can write $\nu = \omega + dd^*\eta + d^*d\eta$ for a harmonic form $\omega \in \mathcal{H}^k X$ and some $\eta \in A^k X$. It suffices now to show that $d^*d\eta = 0$. Since both ν and ω are d-closed we find that

$$0 = dd^*d\eta.$$

Now compute

$$||d^*d\eta||_X = (d^*d\eta, d^*d\eta) = (d\eta, dd^*d\eta) = 0$$

and we are done. □

Remark 156. Let us motivate, briefly, how one might come up with the notion of harmonic forms and these theorems? Consider the following question: can we minimize $||\omega||^2_X$ over all ω in a fixed cohomology class? Every other class is the form $\omega + d\eta$ for some $\eta \in A^{k-1}X$. We consider, for $t \in \mathbb{R}$,

$$f(t) = ||\omega + td\eta||^2_X = ||\omega||^2_X + 2t(\omega, d\eta) + t^2||d\eta||^2_X.$$

We might try to solve $f'(0) = 0$. Taking this derivative we find that $0 = (\omega, d\eta) = (d^*\omega, \eta)$. If ω is to minimize this functional we must have this for each η. But this is equivalent to saying that $d^*\omega = 0$. Of course, we assumed from the beginning that $d\omega = 0$ whence we see harmonic forms appearing.

19.2. **Complex harmonic theory.** To translate this story to the case of complex manifolds we will again have to start with linear algebra. Consider a Hermitian form $h : V \times V \to \mathbb{C}$ on an n-dimensional complex vector space V. Recall that we have

$$h(v, w) = \overline{h(w, v)}.$$

Denote by $V_{\mathbb{R}}$ the underlying real vector space. Multiplication by i on V yields a map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ such that $J^2 = -\operatorname{id}$. Now we obtain a complex vector space of dimension $2n$

$$V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

There is an induced map J on $V_{\mathbb{C}}$ that induces an eigendecomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0} = \ker(J - i\operatorname{id})$ and $V^{0,1} = \ker(J + i\operatorname{id})$, i.e. we have the $\pm i$-eigenspaces, respectively. We make an identification

$$V_{\mathbb{R}} \to V_{\mathbb{C}} \to V^{1,0}$$

sending $v \mapsto 1/2 \cdot (v - iJv)$. This is simply coming from the fact that

$$v = \frac{1}{2}(v - iJv) + \frac{1}{2}(v + iJv).$$

This decomposition induces a decomposition on the wedge products of $V_{\mathbb{C}}$,

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}.$$

Here notice that we are taking the graded vector product.

Recall now that the real part $g = \Re h$ gave us an inner product on $V_{\mathbb{R}}$. Moreover we have that g is compatible with the complex structure in the sense that $g(Jv, Jw) = g(v, w)$ for each v, w. We can write

$$h(v, w) = g(v, w) + ig(v, Jw).$$

Now we might ask how h acts on $V^{1,0}$, for instance. We leave the following as an exercise:

$$h\left(\frac{1}{2}(v-iJv), \frac{1}{2}(w-iJw)\right) = \frac{1}{2}\left(g(v,w)+ig(v,Jw)\right) = \frac{1}{2}h(v,w).$$

Lemma 157. *The decomposition $\Lambda^k V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$ is orthogonal with respect to h.*

Proof. Left as an exercise. □ Do this exercise!

We now turn to the issue of orientations so that we can define a Hodge star. Fix a basis $v_1, \dots, v_n \in V$. Then $V_{\mathbb{R}}$ is naturally oriented with positive basis $v_1, Jv_1, \dots, v_n, Jv_n$. Moreover if the $v_i = e_i$ are an orthonormal basis of V then the resulting basis for the underlying real vector space $e_1, Je_1, \dots, e_n, Je_n$ is a positive orthonormal basis. In this case we define the fundamental element

$$\phi = e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge \cdots \wedge e_n \wedge Je_n.$$

From the real case last time we have a Hodge star operator

$$*_{\mathbb{R}} : \Lambda^k V_{\mathbb{R}} \to \Lambda^{2n-k} V_{\mathbb{R}}.$$

Tensoring with $\mathbb{C}$ we obtain

$$* : \Lambda^k V_{\mathbb{C}} \to \Lambda^{2n-k} V_{\mathbb{C}}$$

which we again call the Hodge star.

Lemma 158. *We have that*

$$\alpha \wedge *\bar{\beta} = h(\alpha, \beta)\phi.$$

Next time we will define the $\bar{\partial}$-Laplacian and discuss harmonic theory for this operator.

20. February 28, 2018

20.1. **Global complex harmonic theory.** We note here some properties of the Hodge star in the complex (local) setting:

(1) we have that $**\alpha = (-1)^k\alpha$ for $\alpha \in \Lambda^k$, i.e. $*^{-1}\alpha = (-1)^k * \alpha$
(2) $*$ sends $V^{p,q}$ to $V^{n-q,n-p}$, again some sort of Poincaré duality

The first point is clear. For the second, start with $0 \neq \beta \in V^{p,q}$. For all $\alpha \in V^{p',q'}$ notice that the orthogonality of the decomposition with respect to h tells us that

$$\alpha \wedge *\beta = h(\alpha, \bar{\beta})\phi$$

is nonzero if and only if $p' = q, q' = p$. Moreover we need $\alpha \wedge *\beta$ to be of type (n,n), so we must have β of type $(n-q, n-p)$ otherwise we find that $\beta = 0$.

We now globalize the above constructions to a compact complex Hermitian manifold (X, h) of (complex) dimension n. We obtain an operator

$$* : A^{p,q}X \to A^{n-q,n-p}X$$

as well as an L^2 inner product

$$(-,-)_X : A^{p,q}X \times A^{p,q}X \to \mathbb{C}$$
$$(\alpha, \beta)_X = \int_X \alpha \wedge *\bar{\beta}.$$

As before the operator $\bar{\partial} : A^{p,q}X \to A^{p,q+1}X$ has a formal adjoint with respect to this metric:

$$(\bar{\partial}\alpha, \beta)_X = (\alpha, \bar{\partial}^*\beta)_X,$$

where $\bar{\partial}^* : A^{p,q} \to A^{p,q-1}$.

Proposition 159. *We can write*

$$\bar{\partial}^* = - * \partial * .$$

Proof. Analagous to the real case. □

Definition 160. The **antiholomorphic Laplacian** is

$$\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : A^{p,q}X \to A^{p,q}X.$$

We say that ω is harmonic if $\bar{\Box}\omega = 0$, and we define

$$\mathcal{H}^{p,q}X = \ker \bar{\Box} \subset A^{p,q}X.$$

Just as before $\bar{\Box}$ is self-adjoint and elliptic, and ω is harmonic if and only if $\bar{\partial}\omega = \bar{\partial}^*\omega = 0$. This induces the obvious map

$$\mathcal{H}^{p,q}X \to H^{p,q}X.$$

Applying the analytic result for elliptic operators as in the real case we find that $\mathcal{H}^{p,q}X$ is finite dimensional and that we have a decomposition

$$A^{p,q}X = \mathcal{H}^{p,q}X \oplus \operatorname{im}(\bar{\Box} : A^{p,q}X \to A^{p,q}X).$$

Theorem 161 (Hodge theorem for (p,q)-forms)**.** *Let (X, h) be a compact Hermitian complex manifold. Then the natural map*

$$\mathcal{H}^{p,q}X \to H^{p,q}X$$

is an isomorphism, i.e. each cohomology class has a unique harmonic representative.

Proof. Analagous to the real case. □

Remark 162. We could equally well have considered the holomorphic Laplacian $\Box$ associated to ∂. It is important to remember moreover that in general there is no connection between Δ and $\bar{\Box}$ (or $\Box$). For instance being $\bar{\Box}$-harmonic does not even necessarily imply being d-closed! In the Kähler case things will be much nicer.

Theorem 163 (Poincaré duality)**.** *Let (X,h) be a compact complex Hermitian manifold. Then the Hodge star operator gives an isomorphism*

$$*: H^{p,q}X \to H^{n-q,n-p}X$$

Proof. Consider $\bar{\partial}^* = - * \partial *$ acting on $A^{p,q}$ with $p+q=k$. We then find

$$\bar{\partial}^* * = -(-1)^k * \partial.$$

We show that if ω is harmonic then $*\omega$ is also harmonic. □

Remark 164. Recall that a map $\phi : V \times W \to k$ is a perfect pairing if for each $v \neq 0$ there exists some $w \in W$ such that $\phi(v,w) \neq 0$. This is equivalent to the fact that the induced mapping $V \to W^*$ is an isomorphism.

Theorem 165 (Kodaira-Serre duality, version 1)**.** *Let X be a compact complex manifold of dimension n. Then*

(1) there is an isomorphism

$$H^n(X, \omega_X) \cong \mathbb{C}$$

where $\omega_X := \Omega_X^n$ is the canonical (line) bundle of X;

(2) there exists a perfect pairing of vector spaces for each p,q

$$H^p(X, \Omega_X^q) \times H^{n-p}(X, \Omega_X^{n-q}) \longrightarrow H^n(X, \omega_X) \cong \mathbb{C}.$$

In particular,

$$H^p(X, \Omega_X^q) \cong H^{n-p}(X, \Omega_X^{n-q})^\vee.$$

Proof. The first statement is a consequence of what we've already proven. Applying Dolbeault and Poincaré duality we have

$$H^n(X, \omega_X) \cong H^{n,n}X \cong H^{0,0}X \cong \mathbb{C}.$$

For the second statement we applying the Dolbeault theorem. We have a map

$$H^{p,q}X \times H^{n-p,n-q}X \xrightarrow{\wedge} H^{n,n}X \xrightarrow{\int} \mathbb{C}$$
$$(\alpha, \beta) \mapsto \alpha \wedge \beta \mapsto \int_X \alpha \wedge \beta$$

To show that this is a perfect pairing, fix $\alpha \neq 0$. Then

$$(\alpha, *\bar{\alpha}) \mapsto \int_X \alpha \wedge *\bar{\alpha} = ||\alpha||_X^2 > 0,$$

and we are done. □

We now have a nice set of invariants for compact complex manifolds.

Definition 166. The **Hodge numbers** of a compact complex manifold X of dimension n are

$$h^{p,q}X := \dim_{\mathbb{C}} H^{p,q}X = \dim_{\mathbb{C}} H^q(X, \Omega_X^p),$$

where the second equality is Dolbeault's theorem.

We note some simple properties of the Hodge numbers:

(1) certainly $h^{p,q}X = 0$ if $p \notin [0,n]$ or $q \notin [0,n]$ by definition;
(2) we have that $h^{p,q}X < \infty$ by PDEs;
(3) we have $h^{n,n}X = h^{0,0}X = 1$ by Poincaré duality;
(4) $h^{p,q}X = h^{n-q,n-p}X = h^{q,p}X = h^{n-p,n-q}X$ by Poincaré and Kodaira-Serre duality.

We will be able to say a lot more in the Kähler case.

21. March 2, 2018

Important correction from last time. Last time we said that the Hodge star yields an isomorphism

$$* : \mathcal{H}_d^{p,q} \xrightarrow{\sim} \mathcal{H}_d^{n-q,n-p}$$

but one has to be careful that we mean harmonic with respect to d, not $\bar{\partial}$!

So the Poincare duality from last time is not quite correct unless we're Kahler?

To show the above equality we show that $*\Delta = \Delta *$. This is straightforwrd using the formulas

$$d^* = (-)^{n(k+1)+1} * d*, \qquad ** = (-1)^{k(n-k)}.$$

Let (X, h) be a compact Kähler manifold. We have the associated closed $(1,1)$-form ω on X, which is not exact (we proved $[\omega] \neq 0 \in H^2(X, \mathbb{R})$ via the Wirtinger theorem).

Definition 167. We introduce the following operators:

(1) the **Lefsheftz operator** is, for all k,

$$L = \omega \wedge - : A^k X \to A^{k+2} X.$$

In particular we have $L : A^{p,q} X \to A^{p+1,q+1} X$.

(2) the adjoint Λ of the Leftshetz operator,

$$g(L\alpha, \beta) = g(\alpha, \Delta\beta)$$

for all α, β.

Lemma 168. *We have the formula*

$$\Lambda = (-1)^k * L*$$

on A^k.

Is this k or $k+2$?

Proof. We fix β, and compute for all α,

$$\begin{aligned} \alpha \wedge *\Lambda\beta &= g(\alpha, \Lambda\beta)\mathrm{vol}(g) \\ &= g(L\alpha, \beta)\mathrm{vol}(g) \\ &= L\alpha \wedge *\beta \\ &= \omega \wedge \alpha \wedge *\beta = \alpha \wedge \omega \wedge *\beta \\ &= \alpha \wedge L * \beta \end{aligned}$$

It follows that $*\Lambda = L*$ and we are done. □

Theorem 169 (Kähler identities)**.** *Let (X, h) be Kähler. Then we have*

$$[\Lambda, \bar{\partial}] = -i\partial^*$$
$$[\Lambda, \partial] = i\bar{\partial}^*$$

Roughly speaking this is true because it holds for the Euclidean metric and then by some sort of osculating result, since there are only first derviatives involved, it holds for any Kähler metric.

Corollary 170. *If X is compact Kähler then $\bar{\Box} = \Box = \Delta/2$.*

Proof. We use the two Kähler identities repeatedly to compute

$$\begin{aligned}\Delta = dd^* + d^*d &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial)\\ &= (\partial + \bar\partial)(\partial^* - i\Lambda\partial + i\partial\Lambda) + (d^* - i\Lambda\partial + i\partial\Lambda)(\partial + \bar\partial)\\ &= \cdots\\ &= \Box - i\bar\partial\Lambda\partial + i\partial\bar\partial\Lambda - i\Lambda\partial\bar\partial + i\partial\Lambda\bar\partial\\ &= \Box + i\partial(\Lambda\bar\partial - \bar\partial\Lambda) + i(\Lambda\bar\partial - \bar\partial\Lambda)\partial\\ &= \Box + \partial\partial^* + \partial^*\partial\\ &= 2\Box,\end{aligned}$$

as desired. Finally notice that $\bar\Delta = \Delta$ which shows that $\bar\Box = \Box$. □

Because of this, if ω is d-harmonic then it is $\bar\partial$-harmonic. Hence we obtain the isomorphism $H^{p,q}X \cong H^{n-q,n-p}X$ that we prematurely arrived at last time.

Corollary 171. *If X is Kähler then $[\Delta, L] = [\Delta, \Lambda] = 0$.*

Proof. Taking adjoints of a Kähler identity we obtain $[\Lambda, \partial]^* = -i\bar\partial$ whence

$$-i\bar\partial = \partial^* L - L\partial^*.$$

Equivalently $L\partial^* = \partial^* L + i\bar\partial$. Now we compute

$$\begin{aligned}L\Box &= L\partial\partial^* + L\partial^*\partial\\ &= \partial L\partial^* + \partial^* L\partial + i\bar\partial\partial\\ &= \partial\partial^* L + i\partial\partial^* + \partial^*\partial L + i\bar\partial\partial\\ &= \Box L,\end{aligned}$$

where we have used that ω is closed, and we are done (since Δ and $\Box$ are the same up to a constant, and for the Λ identity just take adjoints). □

Corollary 172. *If ω is Kähler then ω is harmonic.*

Corollary 173. *Decomposing $\omega \in A^k X$ as $\omega = \sum_{p+q=k} \omega_{p,q}$, then ω is harmonic if and only if all the $\omega_{p,q}$ are harmonic.*

Corollary 174. *The Laplacian repsects the (p, q)-type, i.e. $\Delta A^{p,q} X \subset A^{p,q} X$.*

22. March 5, 2018

22.1. **The Hodge decomposition.** Last time we obtained some applications of the Kähler identities on a Kähler manifold (X, h). Recall the identities were

$$[\Lambda, \bar{\partial}] = -i\partial^*$$
$$[\Lambda, \partial] = i\bar{\partial}^*.$$

Recall that Λ is the adjoint of the Lefshetz operator $L = \omega \wedge -$. We will not prove the Kähler identities in class, though we'll outline the idea. Notice first that the second identity is a conjugate of the first. We notice that the identities involve only partials of order ≤ 1 in coefficients of the metric h. We have seen that h is Kähler if and only if it osculates to order 2 (locally) with respect to the Euclidean metric. Hence it suffices to prove the Kähler identities for the Euclidean metric on $\mathbb{C}^n$, and the theorem reduces to an explicit computation.

Corollary 175. *If X is compact Kähler then for $\alpha \in A^k X$, in the type-decomposition*

$$\alpha = \sum_{p+q=k} \alpha^{p,q} \in A^{p,q} X,$$

we have that α is harmonic if and only if $\alpha^{p,q}$ is harmonic.

Indeed, by the relation between the Laplacians $\Delta = 2\Box = 2\bar{\Box}$ harmonic now has an unambiguous meaning.

Corollary 176. *We have $\mathcal{H}^k X \otimes \mathbb{C} \cong \oplus_{p+q=k} \mathcal{H}^{p,q} X$.*

We now come to the main theorem of Hodge theory.

Theorem 177 (Hodge decomposition)**.** *Let X be a compact Kähler manifold of (complex) dimension n. Then for every k we have a direct sum decomposition*

$$H^k(X; \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is the space of cohomology classes containing a d-closed (p,q)-form. Moreover we have

- $H^{p,q} = \overline{H^{q,p}}$,
- $H^{p,q} \cong H^{p,q} X \cong H^q(X, \Omega_X^p)$.

Notice that this definition of $H^{p,q}$ is independent of the choice of metric h.

Proof. By the previous corollary (after choosing a Kähler metric) we have that

$$H^k(X; \mathbb{C}) \cong \oplus_{p+q=k} \mathcal{H}^{p,q} X \subset H^{p,q}.$$

The other inclusion holds as well. Fix a representative $\alpha \in H^{p,q}$. By the Hodge theorem we can write $\alpha = \omega + \Delta\eta$ where ω is Δ-harmonic. Then

$$\alpha = \omega + dd^*\eta + d^*d\eta$$

from which it follows that $d^*d\eta$ is d-closed, whence zero.

The first listed property is clear. The second property follows from the real Hodge and Dolbeault theorems:

$$H^{p,q} \cong \mathcal{H}^{p,q} X \cong H^{p,q} X \cong H^q(X, \Omega_X^p).$$

□

We now state some immediate corollaries.

Corollary 178. *For X compact Kähler we have $h^{p,q} = h^{q,p}$.*

Corollary 179. *If X is compact Kähler then for each k*

$$b_{2k+1}X \equiv 0 \mod 2.$$

Proof. We have that

$$b_{2k+1}X = \dim_{\mathbb{C}} H^{2k+1}(X;\mathbb{C}) = \sum_{p+q=2k+1} h^{p,q}X = \sum_{p\leq q} 2h^{p,q}.$$

□

Corollary 180. *If X is compact Kähler then for each $k \leq n/2$,*

$$b_{2k}X \neq 0.$$

Proof. We have $[\omega^k] \in H^{2k}(X;\mathbb{C})$ which is not zero by Wirtinger's theorem. □

22.2. **Examples.**

Example 181 (Hopf surface)**.** Consider the surface $X = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ (see earlier for the description for the action). This surface is compact and in fact diffeomorphic to $S^3 \times S^1$. By the Kunneth formula we find that

$$b_0X = b_1X = b_3X = b_4X = 1, b_2 = 0.$$

Hence X is a non-Kähler complex manifold.

The compact Kähler condition gives us nice symmetry properties on the Hodge diamond. We have by the Hodge decomposition and by Serre duality $h^{p,q} = h^{q,p} = h^{n-p,n-q}$. Moreover $\sum_{p+q=k} h^{p,q} = b_kX$. Also the sums of the columns will give you some sort of Hochschild homology.

Example 182 (Compact Riemann surfaces)**.** Let X be a (connected) compact Riemann surface. Then X is obviously Kähler. We have that $h^{0,0} = h^{n,n} = 1$ by duality. The Hodge decomposition tells us that $H^1(X;\mathbb{C}) \cong H^{1,0} \oplus H^{0,1}$. Notice that

$$H^{1,0} \cong H^0(X, \Omega^1_X), \qquad H^1(X, \mathcal{O}_X)$$

Hence $b_1X = 2h^{1,0} = 2h^{0,1}$. Integrally we know that compact surfaces have first Betti number equal to $2g$ where g is the genus. We conclude that $h^{1,0} = h^{0,1} = g$. In particular the Hodge diamond looks like

$$\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

Example 183 (Compact Kähler surfaces)**.** We now move to X compact Kähler with $\dim_{\mathbb{C}} X = 2$. Again, we know that $h^{0,0} = h^{2,2}$. Applying symmetries we have

$$h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} \qquad h^{2,0} = h^{0,2}.$$

We can't say anything about $h^{1,1}$ other than that $h^{1,1} \geq 1$ due to the existence of the Kähler form.

Often people call $h^{1,0} = h^0(X, \Omega^1_X) = q(X)$ the **irregularity** of X. Moreover people call $h^{2,0} = h^0(X, \omega_X)$ the **geometric genus** of X.

Example 184 (Projective space)**.** We recall the singular cohomology of projective space:

$$H^i(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, \text{ even,} \\ 0 & 0 < i < 2n, \text{ odd} \end{cases}$$

Hence $b_k(\mathbb{P}^n) = 1$ for k even and $b_k(\mathbb{P}^n) = 0$ for k odd. This completely determines the Hodge diamond using the fact that $h^{k,k} \geq 1$ due to the presences of the powers of the Kähler form. Hence the Hodge diamond for projective space has 1's on the middle column and zero everywhere else. As an immediate corollary we find that there are no nontrivial global holomorphic forms of any degree $p \geq 1$ on projective spaces.

23. March 7,2018

Here's a quick computation of the singular cohomology of $\mathbb{P}^n$, which is $\mathbb{Z}$ in even degrees (up until $2n$) and zero otherwise. Recall that we can write $\mathbb{P}^n = \mathbb{P}^{n-1} \cup \mathbb{C}^n$. We know that $H_i(\mathbb{C}^n; \mathbb{Z})$ is zero for $i \neq 0$. Write $H = \mathbb{P}^{n-1}$ for the hyperplane. Recall the standard long exact sequence for the cohomology of a pair:

$$\cdots \to H^i(\mathbb{P}^n, H; \mathbb{Z}) \to H^i(\mathbb{P}^n; \mathbb{Z}) \to H^i(\mathbb{P}^{n-1}, \mathbb{Z}) \to H^{i+1}(\mathbb{P}^n, H; \mathbb{Z}) \to \cdots$$

Hence if we understand the relative cohomology we can inductively obtain the cohomology of $\mathbb{P}^n$. To do this we will use Poincaré-Alexander duality:

$$H^i(\mathbb{P}^n, H; \mathbb{Z}) \cong H_{2n-i}(\mathbb{P}^n \setminus H; \mathbb{Z}).$$

We leave the rest as an exercise. Of course there are other ways to compute this such as cellular cohomology.

23.1. More examples.

Example 185 (Complex tori)**.** Let $X = \mathbb{C}^n/\Lambda$ where $\Lambda \subset \mathbb{C}^n$ is a lattice and X is the quotient. If we denote the quotient map by π we obtain a pullback

$$\pi^* : A^{p,q}X \hookrightarrow A^{p,q}\mathbb{C}^n$$

that takes values in forms that are invariant with respect to the action of Λ. Recall that $(\mathbb{C}^n, h_{\mathsf{Eucl}})$ has Kähler form given

$$\omega_{\mathbb{C}^n} = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

This metric and two-form are invariant under the action of the lattice whence we obtain h, ω on X (such that their pullbacks under π are the Euclidean counterparts). We find that $d\omega = 0$ whence X is Kähler. The following very special result characterizes the harmonic forms on X.

Lemma 186. *With respect to the metric h we have that*

$$\Delta(\sum_{JK} \phi_{JK} dz^J \wedge d\bar{z}^K) = \sum_{JK} \Delta\phi_{JK} dz^J \wedge d\bar{z}^K.$$

Proof. Show it for $\mathbb{C}^n$. □

Corollary 187. *We have*

$$\mathcal{H}^{p,q}X = \left\{ \sum_{|J|=p, |K|=q} a_{JK} dz^J \wedge d\bar{z}^K \mid a_{JK} \in \mathbb{C} \right\}.$$

Proof. It suffices to show that harmonic functions on X are constant. This follows of course by usual harmonic theory but we may as well use the fact that $\mathcal{H}^0 X \cong H^0(X; \mathbb{R}) \cong \mathbb{R}$. □

This type of argument is of course due to the fact that we are working on a space locally isometric to $\mathbb{C}^n$.

Corollary 188. *Let $V = H^1(X; \mathbb{C})$. Then*

(1) *We have $V \cong V^{1,0} \oplus V^{0,1}$ with $V^{1,0}$ spanned (over $\mathbb{C}$) by $dz^1, \ldots, dz^n$ and $V^{0,1}$ its conjugate;*

(2) We have $H^i(X;\mathbb{C}) \cong \Lambda^i V$ such that the Hodge decomposition is just

$$\Lambda^i V \cong \oplus_{p+q=i}(\Lambda^p V^{1,0} \otimes \Lambda^p V^{0,1}).$$

Next semester we will discuss how tori give, essentially, weight 1 Hodge structures.

Corollary 189. *It follows that*

$$h^{p,q}X = \dim_{\mathbb{C}} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1} = \binom{n}{p} \cdot \binom{n}{q}$$
$$b_i X = \binom{2n}{i} = \sum_{p+q=i} \binom{n}{p} \cdot \binom{n}{q}.$$

Consider in particular, the case when $n = 1$. The Hodge diamond is just

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

When $n = 2$ we have

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Similarly for $n = 3$.

Remark 190. It turns out that Hodge numbers are constants of deformation-equivalence classes, so they're not very useful for analyzing families, e.g. of tori.

It turns out that Hodge diamond sometimes can be useful to recognize how to decompose a variety into a product or fibration or something like that. This is rather a subtle business, however.

For the following example we will use some facts that we will prove later.

Example 191 (Hypersurface in $\mathbb{P}^n$)**.** Recall that a hypersurface X in $\mathbb{P}^n$ is defined to be $X = Z(F)$ where $F \in \mathbb{C}[X_0, \ldots, X_{n+1}]$. Actually we don't quite know this – all we know is that it's the zero set of an analytic function. We will prove this, however, probably at the beginning of next quarter. Recall that X was Kähler by restricting the Fubini-Study metric and Kähler form on $\mathbb{P}^n$ to X.

Lemma 192. *The restriction*

$$H^k(\mathbb{P}^{n+1};\mathbb{C}) \to H^k(X;\mathbb{C})$$

is injective for all k.

Proof. This is immediate from the fact that the left has dimension only either 0 or 1 and that when it is one, the map is restriction of powers of the Kähler forms. □

The following theorem we will definitely prove in the near future.

Theorem 193 (Weak Lefshetz theorem)**.** *The restriction map in the previous lemma is an isomorphism for each $0 \geq k < n$, whence also for $k > n$ by duality.*

Indeed it will turn out that

$$H^n(X;\mathbb{C}) \cong H^n(\mathbb{P}^{n+1};\mathbb{C}) \oplus H_0^n(X),$$

where H_0 is something we call primitive cohomology. The Hodge decomposition of the whole thing yields a Hodge decomposition of primitive cohomology,

$$H_0^n X \cong H_0^{n,0} X \oplus H_0^{n-1,1} X \oplus \cdots$$

The following theorem is difficult. Or at least subtle.

Theorem 194 (Griffiths)**.** *Let d be the degree of the hypersurface. For every p, there is an isomorphism $H_0^{p,n-p}X \cong R(F)_{(n+1-p)d-(n+2)}$ where $R(F)$ is the Jacobian algebra of F.*

Let us briefly explain $R(F)$: inside $S = \mathbb{C}[x_0, \ldots, x_n]$ we have an ideal

$$J(F) = \langle \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \rangle.$$

It turns out that F itself is in this ideal: $F = 1/d \cdot \sum_{i=0}^n x_i \partial F/\partial x_i$. We define $R(F) = S/J(F)$ which is a graded ring since $J(F)$ is a homogeneous ideal.

Let's look at a smooth curve $X \subset \mathbb{P}^2$, i.e. $X = Z(F)$ where $F \in \mathbb{C}[X_0, X_1, X_2]$ is a homogeneous polynomial of degree d. We have $H^1(\mathbb{P}^2;\mathbb{C}) \cong H^1(X;\mathbb{C}) = 0$. Hence we have

$$H^1(X;\mathbb{C}) = H_0^{1,0}X \oplus H_0^{0,1}X = H^{1,0}X \oplus H^{0,1}X.$$

Let's look at $R(F)$: we have $n = 1$ and $p = 1$ whence

$$H_0^{1,0}X = R(F)_{d-3}.$$

This is great because F has degree d so there can be no relations is $R(F)_{d-3}$ whence

$$H_0^{1,0}X = S_{d-3} = \{\text{monomials of degree } d-3 \text{ in 3 variables}\}$$

$$h^{1,0}X = \binom{d-3+2}{2} = \binom{d-1}{2} - \frac{(d-1)(d-2)}{2}.$$

But this Hodge number is also, by definition, the genus of the curve. Hence we obtain the degree genus formula

$$g = \frac{(d-1)(d-2)}{2}.$$

24. March 9, 2018

Consider $X \subset \mathbb{P}^3$ where $X = Z(F)$ with $F \in S = \mathbb{C}[x_0, \dots, x_3]$ homogeneous of degree 4 (i.e. X is a **K3** surface since $\omega_X = \mathcal{O}_X$). By the weak Lefshetz theorem that we stated last time we find that

$$H^1(X;\mathbb{C}) \cong H^1(\mathbb{P}^3;\mathbb{C}) = 0$$

and similarly for H^3. By the theorem of Griffiths we know that

$$H^2(X;\mathbb{C}) \cong H^2(\mathbb{P}^3;\mathbb{C}) \oplus H_0^2 X.$$

We know that the first summand is $H^{1,1}\,\mathbb{P}^3 \cong \mathbb{C}$. We wish to compute the primitive cohomology $H_0^2 X$ which itself decomposes as $H_0^{2,0}X \oplus H_0^{1,1}X \oplus H_0^{0,2}X$ where the last summand is conjugate to the first. Recall that in general

$$H_0^{p,n-p}X \cong R(F)_{(n+1-p)d-(n+2)}.$$

Here $n = 2, d = 4$. It follows that

$$H_0^{2,0}X \cong R(F)_0 \cong \mathbb{C} \cong H_0^{0,2}$$
$$H_0^{1,1}X \cong R(F)_4 = S_4 / \left(\sum_{i=0}^{3} S_1 \cdot \partial F / \partial x_i \right)$$

We can compute straightforwardly

$$\dim R(F)_4 = \binom{4+3}{3} - 4\binom{1+3}{3} = 19.$$

We conclude that $h^{1,1}X = 20$ whence $b_1 X = b_3 X = 0$ and $b_2 X = 22$.

Remark 195. At some point we will see that the Hodge diamond is invariant in families. Hence we can always pick our favorite hypersurface to do computations with. **Your** favorite hypersurface is the Fermat hypersurface:

$$x_0^d + x_1^d + \cdots + x_n^d = 0.$$

We're going to have three more lectures next week. The plan is to conclude this part with a few theorems of Lefschetz and some sort of introduction on Hodge structures. Next quarter we will go back and do harmonic theory for general vector bundles and get a bunch of theorems from that, such as Kodaira embedding.

24.1. Lefshetz decomposition.

Our aim is to show the "hard Lefschetz theorem," that

$$L^{n-k} : H^k(X;\mathbb{C}) \xrightarrow{\sim} H^{2n-k}(X;\mathbb{C})$$

is an isomorphism. Notice that this is a very concrete duality that is coming from wedging by some form.

Let X be a compact Kähler manifold, $\dim X = n$. We have operators

$$L = \omega \wedge - : A^k X \to A^{k+2} X$$
$$\Lambda : A^k X \to A^{k-2} X$$

and a decomposition

$$A^k X \cong \ker \Lambda \oplus \operatorname{im} L.$$

Lemma 196. *We have that* $H = [L, \Lambda] = (k - n)\,\mathrm{id}$.

Proof. Left as an exercise. Hint: compute it for $\mathbb{C}^n$ and then extend to all Kähler manifolds via the osculation result. $\square$

Definition 197. We say that $\gamma \in A^k X$ is **primitive** if $\Lambda\gamma = 0$.

By the decomposition mentioned above any form can be decomposed

$$\alpha = \alpha_0 + L\beta_0$$

where α_0 is primitive and $\beta_0 \in A^{k-2}X$. We can repeat this procedure by writing

$$\beta_0 = \alpha_1 + L\beta_1$$

where α_1 is primitive and $\beta_1 \in A^{k-4}X$, and so on. Hence for each form we obtain a decomposition

$$\alpha = \alpha + L\alpha_1 + L^2\alpha_2 + \cdots + L^{[k/2]}\alpha_{[k/2]}$$

where $\alpha_j \in A^{k-2j}$ is primitive. It is not yet clear that this decomposition is unique.

Notice that there are no primitive forms of higher than middle degree.

Lemma 198. *If a form $\alpha \in A^{n-l}X$ is primitive then for each $k \geq 1$ we have*

$$\Lambda L^k\alpha = k(l-k+1)L^{k-1}\alpha.$$

Moreover, if $l < 0$ then $\alpha = 0$.

Proof. We know that $\Lambda\alpha = 0$ whence

$$H\alpha = -l\alpha.$$

We will prove the first statement by induction. For $k = 1$ it is clear by the equation above. Now suppose we know the statement for k.

$$\begin{aligned}
\Lambda L^{k+1}\alpha &= \Lambda L L^k\alpha = (L\Lambda - H)L^k\alpha \\
&= L\Lambda L^k\alpha - HL^k\alpha \\
&= k(l-k+1)L^k\alpha - (-l+2k)L^k\alpha \\
&= (kl - k^2 + k + k - 2k)L^k\alpha = (k+1)(l-k)L^k\alpha.
\end{aligned}$$

Now suppose that $l < 0$. Write

$$k_0 = \min\{k \mid L^k\alpha = 0\}.$$

Such a minimum exists because certainly $L^{n+1}\alpha = 0$. Hence

$$0 = \Lambda L^{k_0}\alpha = k_0(l-k_0+1)L^{k_0-1}\alpha.$$

If $k_0 \neq 0$ we conclude that $L^{k_0-1}\alpha = 0$ which is a contradiction. Hence $k_0 = 0$. $\square$

Hence we actually know a bit more about this decomposition.

Proposition 199. *For all $\alpha \in A^k X$ we have a decomposition*

$$\alpha = \alpha_0 + L\alpha_1 + \cdots = \sum_{j=\max\{k-n,0\}}^{[k/2]} L^j\alpha_j$$

with $\alpha_j \in A^{k-2j}$ primitive. Moreover this decomposition is unique.

Proof. If $L^j\alpha_j \neq 0$ then $j \geq k-n$: indeed, apply the lemma above with $l = n-k+2j$ whence $j \geq n-k+2j$. To prove uniqueness assume that $\alpha = 0$. Then we wish to show that $\alpha_j = 0$ for all j. Apply Λ to the sum in the statement:

$$\begin{aligned} 0 &= \Lambda\alpha_0 + \Lambda L\alpha_1 + \cdots \\ &= \sum_j j(j+n-k+1)L^{j-1}\alpha_j \end{aligned}$$

from which we see that $\alpha_1 = L(\cdots)$. But $\alpha_1 \in \ker\Lambda$ since it is primitive and $\alpha_1 \in \operatorname{im} L$ by this argument. Hence by the direct sum decomposition $\alpha_1 = 0$. Applying Λ again and conclude similarly for α_2, etc. $\square$

Roughly speaking the primitive forms generate all forms under the Lefshetz operator.

Theorem 200 (Lefshetz decomposition)**.** *If X is a compact Kähler manifold with Kähler form ω, and L, Λ are the induced operators, then each $\alpha \in H^k(X;\mathbb{C})$ admits a unique decomposition*

$$\alpha = \sum_{j=\max\{k-n,0\}}^{[k/2]} L^j\alpha_j$$

with $\alpha_j \in H^{k-2j}(X;\mathbb{C})$ primtive. Moreover, this decomposition is compatible with the Hodge decomposition.

Proof. This is immediate: every class is represented by a harmonic form and L or Λ applied to harmonic forms again yields harmonic forms (by the Kähler condition). For the last statement we notice that $L(A^{p,q}) \subset A^{p+1,q+1}$ and $\Lambda(A^{p,q}) \subset A^{p-1,q-1}$. If $\alpha = \sum \alpha^{p,q}$ is primitive then each $\alpha^{p,q}$ must be primitive, and if $\alpha \in A^{p,q}$ then $\alpha_j \in A^{p-j,q-j}$. $\square$

Next time we will see how this implies the hard Lefschetz theorem and we will discuss the Hodge-Riemann bilinear relations.

25. March 12, 2018

25.1. **Hard Lefschetz theorem.** Recall the Lefschetz theorem from last time. We use it to prove the following.

Corollary 201 (Hard Lefschetz theorem)**.** *Let X be a compact Kähler manifold. The map*

$$L^{n-k} : H^k(X;\mathbb{C}) \to H^{2n-k}(X;\mathbb{C})$$

is an isomorphism for each $0 \leq k \leq n$.

Proof. Surjectivity is easy: let $\alpha \in H^{2n-k}(X;\mathbb{C})$ and apply the Lefschetz decomposition:

$$\alpha = \sum_{j=n-k}^{[n-k/2]} L^j \alpha_j = L^{n-k}(\cdots).$$

Next we prove injectivity. Assume that $L^{n-k}\beta = 0$ for some $\beta \in H^k(X;\mathbb{C})$. Apply the Lefschetz decomposition and then apply L^{n-k} (write $i = n-k+j$):

$$0 = L^{n-k}\beta = \sum_{j=\max\{k-n,0\}}^{[k/2]} L^{n-k+j}\beta^j = \sum_{n-k}^{[n-k/2]} L^i \beta_{i+k-n}.$$

But this is just the Lefschetz decomposition for a class in $H^{2n-k}(X;\mathbb{C})$. By uniqueness of the Lefschetz decomposition we find that each $\beta_j = 0$. $\square$

25.2. **Some representation theory.** Actually all of these results of Lefschetz follow by straightforwardly applying the representation theory of $\mathfrak{sl}_2\mathbb{C} = \mathfrak{sl}_2 = \{A \in M_{2,2}\mathbb{C} \mid \operatorname{tr} A = 0\}$. This vector space is a 3-dimensional Lie algebra under the usual commutator of matrices. There is a distinguished set of generators for this Lie algebra:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to compute the relevant commutators:

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H.$$

Definition 202. A (complex) finite-dimensional representation V of $\mathfrak{sl}_2$ is a Lie algebra homomorphism

$$\rho : \mathfrak{sl}_2 \to \operatorname{End} V.$$

In other words,

$$\rho([A,B]) = [\rho(A), \rho(B)] = \rho(A)\rho(B) - \rho(B)\rho(A).$$

Equivalently, a representation is the data of three operators in $\operatorname{End} V$ satisfying the commutation relations that E, F, H do in $\mathfrak{sl}_2$.

Lemma 203. *If X is a compact Kähler manifold then the operators $L, \Lambda, H = [L, \Lambda]$ define a representation of $\mathfrak{sl}_2$ on $V = H^*(X;\mathbb{C}) = \oplus_{k \geq 0} H^k(X;\mathbb{C})$.*

Proof. Notice that by definition $H = [L, \Lambda]$ and moreover that $H = (k-n)\,\mathrm{id}$ on k-forms (this is where we crucially use the Kähler property). Then for $\alpha \in A^k X$,

$$\begin{aligned}
[H,L]\alpha &= HL\alpha - LH\alpha = H(\omega \wedge \alpha) - \omega \wedge H\alpha \\
&= (k+2-n)\omega \wedge \alpha - (k-n)\omega \wedge \alpha \\
&= 2\omega \wedge \alpha.
\end{aligned}$$

Similarly for $[H, \Lambda]$. □

Let's state a few facts about the basics of representation theory (c.f. Griffiths-Harris, p.119-120):

(1) every representation $\rho : \mathfrak{sl}_2 \to \operatorname{End} V$ decomposes into a direct sum of irreducible representations;
(2) all eigenvalues of H are integers and V decomposes as a direct sum of one-dimensional eigenspaces of H,
$$V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-m+2} \oplus V_{-m} =: V(m)$$
for some $m \geq 0$;
(3) if ρ is irreducible then V is generated by a primitive vector, v, i.e. $Fv = 0$, and v is an eigenvector for H. Moreover a basis for v is given $v, Ev, \dots E^\ell v$ where $Hv = -\ell v$. For each j, $E^j v$ is an eigenvector for H with eigenvalue $-\ell + 2j$;
(4) combining the previous two statements we see that the irreducible representations of $\mathfrak{sl}_2$ are in one-to-one correspondence with $\mathbb{N}$ where $m \in \mathbb{N}$ corresponds to the $(m+1)$-dimensional representation $V(m)$.

By the way, a great reference for this in the context of Hodge theory is Wells' book.

Some diagrams here that I didn't draw explaining how to relate these facts to the Lefschetz decomposition.

Insert some cryptic comment here about how the $\mathfrak{sl}_2$ representation theory tells you that there are no more interesting dualities/identities to be found from the Kähler structure.

Remark 204. Let $0 \neq \alpha \in A^{p,q}X$ be primitive. Last time we showed that $L^{n-k}\alpha \neq 0$ but $L^{n-k+1}\alpha = 0$. Notice that $L^{n-k}\alpha$ is of type $(n-q, n-p)$. There is another form of the same degree: $*\alpha$.

Lemma 205. *If $\alpha \in A^{p,q}X$ is primitive then*
$$*\alpha = (-1)^{k(k+1)/2} \frac{i^{p-q}}{(n-k)!} L^{n-k}\alpha$$

Proof. Reduce to the Euclidean case and do the computation there. Alternatively it follows from some representation theory of $\mathfrak{sl}_2$. □

Definition 206. For each $k \leq n$ define the bilinear form
$$Q : A^k X \times A^k X \to \mathbb{R}$$
by
$$Q(\alpha, \beta) := (-1)^{k(k-1)/2} \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Notice that
$$Q(\alpha, \beta) = (-1)^k Q(\beta, \alpha)$$
whence Q is symmetric or antisymmetric depending on the parity of k.

If $\alpha, \beta \in A^{p,q}X$ with $p+q=k$ are primitive forms then the above lemma shows

$$\begin{aligned}(\alpha,\beta)_X &= \int_X \alpha \wedge *\bar{\beta} \\ &= (-1)^{k(k+1)/2}\frac{i^{q-p}}{(n-k)!}\int_X \alpha \wedge L^{n-k}\bar{\beta} \\ &= \frac{i^{p-q}}{(n-k)!}Q(\alpha,\bar{\beta}).\end{aligned}$$

Next time we will use this to show the Hodge-Riemann bilinear relations.

26. March 14, 2018

26.1. **Hodge-Riemann bilinear relations.** Recall the bilinear form Q defined last time.

Theorem 207 (Hodge-Riemann bilinear relations)**.** *The bilinear form Q has the following properties:*

(1) in the Hodge decomposition

$$H^k(X;\mathbb{C}) = \oplus_{p+q=k} H^{p,q}X,$$

the subspaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal with respect to Q unless $p = q'$ and $q = p'$;

(2) if $0 \neq \alpha \in H^{p,q}X$ is primitive then

$$i^{p-q}Q(\alpha,\bar{\alpha}) > 0.$$

Proof. The first point follows from type considerations. For the second point, we compute, by the the remark at the end of last class,

$$i^{p-q}Q(\alpha,\bar{\alpha}) = (n-k)!\|\alpha\|_X^2 > 0.$$

□

Example 208. Let X be a compact Riemann surface. Then $H^{1,0}X = H_0^{1,0}X \subset H^1X$, as we've seen. We compute, for $\alpha \in H_0^{1,0}X$ real,

$$\begin{aligned} iQ(\alpha,\bar{\alpha}) &= \int_X \alpha \wedge *\alpha = \int_X \alpha \wedge i\bar{\alpha} \\ &= i\int \alpha \wedge \bar{\alpha}. \end{aligned}$$

Locally if we write $\alpha = f(z)dz$ on an open U then we have

$$i\int_U |f(z)|^2 dz \wedge d\bar{z} = 2\int_X |f(z)|^2 dxdy.$$

Hence we can check positivity directly, without the bilinear relations.

Example 209. Let X be a compact Kähler surface. Then

$$H^2(X;\mathbb{C}) = H^{2,0}X \oplus H^{1,1}X \oplus H^{0,2}X$$

where the first and last terms are primitive. We have that

$$H^{1,1}X = \mathbb{C}[\omega] \oplus H_0^{1,1}X$$

since it must decompose as the image of L and the kernel of Λ.

On $H^{2,0}X \oplus H^{0,2}X$ we have that $Q(\alpha,\alpha) = -\int_X \alpha\wedge\alpha$. Define $\tilde{Q} = -Q$. Then (assume α is a form plus its conjugate, then extend by linearity)

$$\begin{aligned} \tilde{Q}(\alpha,\alpha) &= \int_X (\alpha^{2,0}+\alpha^{0,2}) \wedge (\alpha^{2,0}+\alpha^{0,2}) \\ &= 2\int_X \alpha^{2,0}\wedge\alpha^{0,2} = 2\int_X \alpha^{2,0}\wedge\overline{\alpha^{2,0}} \\ &= 2\|\alpha^{2,0}\|^2 > 0. \end{aligned}$$

What about a primitive $(1,1)$-form α? We have

$$\tilde{Q}(\omega, \alpha) = \int_X \omega \wedge \alpha = 0$$

since $L\alpha = \omega \wedge \alpha$ is a primitive 4-form. Next, note that

$$\tilde{Q}(\omega, \omega) = \int_X \omega \wedge \omega = \mathrm{vol} X > 0.$$

Finally, what about $H^{1,1}X$ with itself? Suppose that $\alpha \in H^{1,1}X$ is real,

$$\tilde{Q}(\alpha, \alpha) = \int_X \alpha \wedge \bar{\alpha}) = -Q(\alpha, \bar{\alpha}),$$

which is negative by the Hodge-Riemann bilinear relations.

Corollary 210 (Hodge index theorem)**.** *If X is a compact Kähler surface, the* ***intersection pairing***

$$H^2(X; \mathbb{R}) \times H^2(X; \mathbb{R}) \to \mathbb{R}$$

that sends $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$ has index

$$(2h^{2,0}X + 1, h^{1,1}X - 1).$$

On $H^{1,1}X$ it has index

$$(1, h^{1,1}X - 1).$$

More generally, let X be a compact Kähler manifold of dimension $n = 2k$ and consider the analogous $\tilde{Q} : H^{2k}(X; \mathbb{R}) \times H^{2k}(X; \mathbb{R}) \to \mathbb{R}$. Then define the signature $\mathrm{sgn}(X) = \mathrm{sgn}(\tilde{Q})$ of X.

Theorem 211. *We have that sgn$(X) = \sum_{p,q}^{2k} (-1)^p h^{p,q} X$.*

26.2. **Pure Hodge structures.**

Definition 212. An **integral (rational) Hodge structure of weight** k is a finitely generated abelian group $H_{\mathbb{Z}}$ (or Q-vector space $H_{\mathbb{Q}}$) together with a direct sum decomposition

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=k} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$ (and similarly for the rational case).

(at this point there was news of someone with a gun on campus so we left off here)

27. March 16, 2018

27.1. **Hodge structures.** Recall our definition of (pure) integral or rational Hodge structure is a finitely generated free abelian group $H = H_{\mathbb{Z}}$ (or finite dimensional $\mathbb{Q}$-vector space $H = H_{\mathbb{Q}}$) together with a direct sum decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Z},\mathbb{Q}} \mathbb{C} = \oplus_{p+q=k} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$.

Example 213. The kth cohomology $H^k(X;\mathbb{Z})$ for X a compact Kähler manifold is a Hodge structure of weight k by the Hodge decomposition theorem. Recall that this is independent of the choice of Kähler structure.

Example 214. If ω a Kähler form is such that $[\omega] \in H^2(X;\mathbb{Q}) \subset H^2(X;\mathbb{R})$ then $H^k_0(X;\mathbb{Q})$ is a $\mathbb{Q}$-Hodge structure of weight k.

Example 215. If V is a $\mathbb{Q}$-vector space such that $V_{\mathbb{R}}$ has an (almost) complex structure J then recall that $\Lambda^k V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$.

Example 216 (Tate twist)**.** Write $\mathbb{Z}(k)$ for the unique Hodge structure of dimension 1 and weight $-2k$. Similarly for $\mathbb{Q}$. In other words $\mathbb{C}(k) = H^{-k,-k}$.

For instance we saw that $H^{2k}(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}(-k)$.

It turns out that it is generally better to think about Hodge *filtrations*, because they vary holomorphically in families of complex manifolds, whereas the $H^{p,q}$ don't.

Definition 217. With H as above, we define

$$F^m H_{\mathbb{C}} := \oplus_{p \geq m} H^{p,q}$$

This gives us a **Hodge filtration**

$$F^k H_{\mathbb{C}} \subset \cdots \subset F^1 H_{\mathbb{C}} \subset F^0 H_{\mathbb{C}} = H_{\mathbb{C}}.$$

Remark 218. Notice that the Hodge filtration determines the Hodge structure:

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^{k-p} H_{\mathbb{C}}}.$$

Moreover

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{k-p+1} H_{\mathbb{C}}}.$$

Example 219. Give H and H' Hodge structures of weights k and l. Then $H \otimes_{\mathbb{Z}} H'$ is a Hodge structure of weight $k+l$. Indeed,

$$(H \otimes_{\mathbb{Z}} H') \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\mathbb{C}} \oplus_{\mathbb{C}} H'_{\mathbb{C}} = \oplus_{p+q=k+l} (H \otimes H')^{p,q}$$

where $(H \otimes H')^{p,q} = \oplus_{p'+p''=p,q'+q''=q} H^{p,q} \otimes H'^{p',q'}$.

If H is a Hodge structure of weight k then we define

$$H(\ell) := H \otimes_{\mathbb{Z}} \mathbb{Z}(\ell),$$

the Tate twist of H, of weight $k - 2l$.

More geometrically, given X, X' compact Kähler manifolds, the Künneth formula for singular cohomology tells us that

$$H^k(X \times X;\mathbb{Z}) \cong \oplus_{r+s=k} H^r(X;\mathbb{Z}) \otimes H^s(X';\mathbb{Z}).$$

Notice that this is a *decomposable* Hodge structure.

Consider, for instance, consider

$$H^k(X \times \mathbb{P}^1; \mathbb{Z}) \cong H^k(X; \mathbb{Z}) \oplus H^{k-2}(X; \mathbb{Z}) \otimes H^2(\mathbb{P}^1; \mathbb{Z})$$
$$= H^k(X; \mathbb{Z}) \oplus H^{k-2}(X; \mathbb{Z})(-1)$$

where we use the notation of a Tate twist.

So far these notions have been a completely obvious formalization of cohomology of compact Kähler manifolds. Let's do something not so obvious.

Definition 220. An **integral polarized Hodge structure of weight** k is a $\mathbb{Z}$-Hodge structure H together with an intersection form

$$\underline{Q} : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z},$$

a bilinear form satisfying the following properties:

(1) $\underline{Q}$ is symmetric if k is even and anti-symmetric if k is odd;
(2) if
$$S(\alpha, \beta) := i^k \underline{Q}(\alpha, \bar{\beta})$$
(induced on the complexification) then the decomposition of H is orthogonal with respect to S;
(3) we have that
$$i^{p-q-k}(-1)^{k(k-1)/2} S(\alpha, \bar{\alpha}) > 0$$
for $0 \neq \alpha \in H^{p,q}$.

The rational case is analogous.

This notion is important because in the following example $[\omega]$ is a rational class if and only if X compact Kähler is actually projective.

Example 221. Let X is a compact Kähler manifold. If in addition $[\omega] \in H^2(X; \mathbb{Q})$ then the primitive cohomology $H = H_0^k(X; \mathbb{Q})$ carries a polarized Hodge structure of weight k. Recall that

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \omega^{n-k} \wedge \alpha \wedge \beta$$

and

$$\underline{Q}(\alpha, \beta) = (-1)^{k(k-1)/2} Q(\alpha, \beta).$$

The first property above follows from last class. The second property follows from the orthogonality property of Q again discuss last class. Finally the last property follows from the Hodge-Riemann bilinear relations.

In fact if one is careful about signs then you can put these forms together using the Lefschetz decomposition to get a polarized structure on the whole cohomology.

This gives a sense in which algebraic varieties among compact Kähler manifolds are as rare as rational points in a real vector space.

Definition 222. A morphism $\rho : H_{\mathbb{Z}} \to G_{\mathbb{Z}}$ of abelian groups between Hodge structures of weight k and $k + 2r$ (for $r \in \mathbb{Z}$), respectively, is a **morphism of Hodge structures of type** (r, r) if

$$\phi \otimes \mathrm{id} = \phi H^{p,q} \subset G^{p+r,q+r}$$

for each p, q. It is easy to check that this is true if and only if $\phi(F^s H_{\mathbb{C}}) \subset F^{s+2r} H_{\mathbb{C}}$ for all s.

Lemma 223. *Morphisms of Hodge structures are in fact strict, i.e.*

$$\operatorname{im}\phi \cap F^{s+r}G_{\mathbb{C}} = \phi(F^s H_{\mathbb{C}}).$$

Proof. Straightforward exercise. □

Proposition 224. *If $H_{\mathbb{Q}}$ is a Hodge structure of weight k then the data of a polarization $\underline{Q} : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \times \mathbb{Q}$ is equivalent to the data of a morphism of Hodge structures of type (0,0) $H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}(-k)$ such that the third condition is satisfied.*

Proof. Straightforward exercise. □

Proposition 225. *There is a one-to-one correspondence between isomorphism classes of integral Hodge structures of weight 1 and isomorphism classes of complex tori.*

Later we will see that there is a similar correspondence for polarized such and projective tori (abelian varieties)

Proof. Let me just give you the maps. Given a Hodge structure of weight 1 H we have $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$. Recall that $H_{\mathbb{R}} \hookrightarrow H_{\mathbb{C}}$ is identified with $H^{1,0}$. But $H \hookrightarrow H_{\mathbb{R}}$ is a lattice, then, inside $H^{1,0}$. But $H^{1,0}/H$ exactly gives us a torus. Conversely, if T is torus, it certain has a weight one Hodge structure $H = H^1(T;\mathbb{Z})$. How do we recover the torus from the Hodge structure? It turns out that

$$T \cong (H^{1,0})^\vee/\Gamma, \qquad \Gamma = H^1(T;\mathbb{Z})^\vee,$$

where Γ is the dual lattice. □

28. April 3, 2018

Our first main goal for the second part of this class are:

(1) Kodaira vanishing and embedding
(2) weak Lefschetz theorem
(3) Hodge classes
(4) Chow's theorem
(5) families of varieties and deformation theory

28.1. **Why do we need connections?** Let X be a complex manifold of dimension n and $\pi : E \to X$ be a holomorphic vector bundle on X of rank r. We want to generalize the theory of differential forms in the case where we twist by a vector bundle.

Definition 226. Let $\pi : E \to X$ be a holomorphic vector bundle. Denote by $A^0(U, E)$ the C^∞-sections of E over U and by $A^{p,q}(U, E)$ the C^∞-forms of type (p, q) with values in E. In other words, these are locally form-valued vectors that glue according to the transition functions of E. Equivalently, these are just the sections of the sheaf $\mathcal{A}_X^{p,q} \otimes E$ on U. We denote the resulting sheaves as $\mathcal{A}^{p,q}(E)$, etc.

For instance, consider $\omega_i \in A^{p,q}(U_i)^{\oplus r}$ such that $g_{ij}\omega_j = \omega_i$ on $U_i \cap U_j$.

Example 227. The simplest example is that of $r = 1$ where $E = L$ is a line bundle. Then $g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$. Now $\omega \in \Gamma(X, \mathcal{A}^{p,q}(L))$ is the data of one-forms $\omega_i \in \mathcal{A}^{p,q}(U_i)$ such that $g_{ij}\omega_j = \omega_i$.

We can define, now,

$$\bar\partial : A^0(X, E) \to A^1(X, E)$$

as: for $s \in A^0(X, E)$ choose a trivialization and consider the resulting $s_{i,j} : U_i \to \mathbb{C}$. Take locally

$$\bar\partial s = (\bar\partial s_{i,1}, \dots, dbs_{i,r}) \in A^{0,1}(U_i)^{\oplus r}$$

and notice that on overlaps

$$g_{ij}s_j = s_i$$

so differentiating

$$\bar\partial s_i = \bar\partial(g_{ij}s_j) = g_{ij}\bar\partial s_j$$

where here we are differentiating each component of the vector or matrix. Hence this definition glues. Notice that we cannot do the same thing for d or ∂. This flaw is what leads us to the notion of connections.

28.2. **Connections.**

Definition 228. A **connection on** E (or a covariant derivative) is a map

$$\nabla : \Gamma(X, T_X) \times A^0(X, E) \to A^0(X, E)$$

sending $(\xi, s) \mapsto \nabla_\xi s$, satisfying

(1) ∇ is $A^0(X)$-linear in ξ;
(2) ∇ is $\mathbb{C}$-linear in s;
(3) the Leibniz rule

$$\nabla_x(f \cdot s) = (\xi \cdot f)s + f\nabla_\xi s.$$

As usual, we can think of connections differently in terms of local coordinate data. If we fix trivializations we obtain a natural frame $s_j(x)$ on a trivialization open U (given by the usual basis for $\mathbb{C}^n$). We define one-forms θ_{jk} on U by the action of the connection on this frame:

$$\nabla_\xi s_j = \sum_{k=1}^{r} \theta_{jk}(\xi) s_k.$$

Notice that these one-forms determine the connection ∇ completely. We will often use the shorthand

$$\nabla s_j = \sum \theta_{jk} \otimes s_k.$$

In other words we can think of a connection as a map

$$\nabla : A^0(X, E) \to A^1(X, E)$$

with the corresponding properties.

To define, in a natural way, the various differential operators we want, on forms twisted by any vector bundle we will need to fix a metric. Recall that a Hermitian metric h on E is a collection of Hermitian inner products $h_x : E_x \times E_x \to \mathbb{C}$ varying smoothly x, i.e. given sections $s_1, s_2 \in A^0(X, E)$, then $h(s_1, s_2) = h_x(s_1(x), s_2(x)) : X \to \mathbb{C}$ is a smooth function. In a fixed trivialization we have a standard frame $s_1, \ldots, s_r$ and h is determined by $h(s_j, s_k) = h_{jk}$ which satisfies $h_{kj} = \overline{h_{jk}}$ and that $(h_{jk})_{j,k}$ is a positive-definite matrix at each x.

Recall that we have a decomposition $T_{\mathbb{C}}X = T'X \oplus T''X$ whence by complexification, a connection ∇ on the real tangent bundle of X splits into $\nabla \otimes 1 = \nabla' + \nabla''$ connections on the holomorphic and antiholomorphic tangent bundles.

Proposition 229. *Fix a holomorphic vector bundle $E \to X$ with a Hermitian metric h. Then there exists a unique connection ∇ on E such that*

(1) ∇ is compatible with the metric, i.e.

$$\xi \cdot h(s_1, s_2) = h(\nabla_\xi s_1, s_2) + h(s_1, \nabla_\xi, s_2);$$

(2) and ∇ is compatible with the complex structure, i.e.

$$\nabla''_\xi s = (\bar\partial s)(\xi),$$

i.e. $\nabla'' = \bar\partial$. Here $\bar\partial s$ is as defined above.

We call this connection the Chern connection associated to (E, h). Notice that the splitting of ∇ here is given by the splitting of the complexification of the real tangent bundle of X.

Remark 230. If ∇ is the Chern connection, we will think of it as our usual differential d. Indeed, when E is just the trivial line bundle (with the Euclidean metric), the Chern connection $\nabla = d$ and $\nabla' = \partial$ and $\nabla'' = \bar\partial$.

Next time we will show that locally, for the Chern connection,

$$\theta_{jk} = \sum_{l=1}^{r} h^{lk} \partial h_{jl}$$

where $(h^{lk}) = (h_{lk})^{-1}$. Writtten as matrices, $\theta_{jk} = \partial h \cdot h^{-1}$.

29. April 4, 2018

29.1. **Connections, continued.** We prove the existence statement from last time.

Proof. We work locally, with respect to a trivialization $\phi : \pi^{-1}U \to U \times \mathbb{C}^r$. We have a natural local frame $s_1, \ldots, s_r$ coming from the usual basis for $\mathbb{C}^r$. Any connection ∇ is determined by its connection one-forms:

$$\nabla s_j = \sum_{k=1}^r \theta_{jk} s_k.$$

We write the matrix of the metric locally as $h_{jk} = h(s_j, s_k)$. For ∇ to be compatible with the complex structure we must have that

$$\nabla'' s_j = \bar{\partial} s_j = 0$$

since s_j is a holomorphic section. We conclude that we must have $\theta_{jk} \in A^{1,0}U \subset A^1 U$.

Next we check what compatibility with the metric forces on us:

$$dh_{jk} = h(\nabla s_j, s_k) + h(s_j, \nabla s_k).$$

If we insert the expression for ∇s_j above, we find

$$(\partial + \bar{\partial}) h_{jk} = \sum_{l=1}^r h_{lk}\theta_{jl} + h_{jl}\overline{\theta_{kl}}$$

whence by the previous paragraph, comparing types, we get

$$\partial h_{jk} = \sum_l h_{lk}\theta_{jl} \qquad \bar{\partial} h_{jk} = \sum_l h_{jl}\overline{\theta_{kl}}.$$

Writing $(h^{pq}) = (h_{pq})^{-1}$, we find that

$$\theta_{jk} = \sum_{l=1}^r h^{lk} \partial h_{jl}.$$

Hence there exists a unique local solution. Gluing, we obtain a unique global solution. □

29.2. **Line bundles.** Let's specialize to the case where $r = 1$. The data of a line bundle is equivalent to the data of a trivializing open cover with a transition Čech cocycle $g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$. Recall that this immediately gives us a group isomorphism between the isomorphism classes of line bundles and $H^1(X, \mathcal{O}_X^\times)$ (really here we mean the Čech cohomology, but if we assume that our cover is nice enough this computes sheaf cohomology). Notice that the group operation on line bundles is given by the tensor product (the inverse is given by reciprocating the transition cocycle). In other words,

$$\operatorname{Pic} X \cong H^1(X; \mathcal{O}_X^\times).$$

Example 231. Suppose we fix an analytic hypersurface $D \subset X$, i.e. the data of (U_i, f_i) such that $D \cap U_i = Z(f_i)$. We define the line bundle $\mathcal{O}_X(-D)$ to be given by this open cover and the transition functions $g_{ij} = f_j/f_i$, which is in $\mathcal{O}_X^\times(U_i \cap U_j)$ since $Z(f_i) = Z(f_j)$ (here we might have to modify the f_i slightly e.g. so that there are no squares, etc). The inverse $L^{-1} = \mathcal{O}_X(D)$ has transition functions $g_{ij} = f_i/f_j$, which is the line bundle associated to D.

Notice that there exists a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Indeed, sections of $\mathcal{O}_X(-D)(U)$ are functions vanishing on $U \cap D$. Notice that the first map is between line bundles the map itself is not as a map of vector bundles as it is not constant rank (it is nonzero outside of D!).

Let us examine more closely the Chern connection from above in the case of line bundles. Notice that a local frame on U is simply the data of a nonvanishing section which we may think of as a function $s : U \to \mathbb{C}$. The metric in this case is defined just by a function $h(x) = h(s(x), s(x)) \in \mathbb{R}_{>0}$, instead of a matrix. The Chern connection is now given by a single one-form,

$$\nabla s = \theta \otimes s$$

for $\theta \in A^{1,0}U$. The formula from the previous proposition is written

$$\theta = h^{-1}\partial h = \partial \log h.$$

Definition 232. We define $\Theta = \bar{\partial}\theta = -\partial\bar{\partial}\log h \in A^{1,1}U$ to be the **curvature of** ∇.

Proposition 233. *Let (L, h) be as above. Then the local curvatures on U glue to a global two-form that we again call the curvature of ∇. Moreover*

$$\partial\Theta = \bar{\partial}\Theta = 0,$$

and the resulting cohomology class $[\Theta] \in H^{1,1}X$ is independent of the metric. Finally, with the induced metric on L^{-1} we have that the curvature of L^{-1} is $-\Theta$ and the curvature of the tensor product of two line bundles is the sum of the curvatures.

Proof. To check that the curvature globalizes, we choose an overlapping trivialization $s' : U' \to \mathbb{C}$. On the intersection we have that $s = fs'$ for $f \in \mathcal{O}_X(U \cap U')$. Now $h(s', s') = |f|^2 h(s, s)$ so

$$\begin{aligned}\Theta' &= -\partial\bar{\partial}\log h(s', s') = -\partial\bar{\partial}\log|f|^2 - \partial\bar{\partial}\log h(s, s) \\ &= \Theta.\end{aligned}$$

The vanishing of the first term is a computation we did last quarter.

Next we check that $[\Theta] \in H^{1,1}X$ is independent of the choice of h. Fix another metric h'. We can write $h' = \psi h$ for some function $\psi : X \to \mathbb{R}_{>0}$. We compute

$$\Theta' = -\partial\bar{\partial}\log h' = -\partial\bar{\partial}\log\psi - \partial\bar{\partial}\log h = \Theta + \bar{\partial}\partial\log\psi,$$

as desired.

We leave the rest of the proof as an exercise. □

Remark 234 (Curvature for vector bundles in general). In general, curvature is a 2-form valued section of a certain vector bundle. Recall that we can think of the connection as $\nabla : A^0(X, E) \to A^1(X, E)$ This operator uniquely extends to $\nabla : A(X, E) \to A(X, E)$ that raises the differential form degree by 1:

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|} \wedge \nabla s.$$

By definition, the curvature is defined to be ∇^2, which turns out to be a two-form valued global section of the vector bundle $\operatorname{End} E$. Of course, one should check that ∇^2 is tensorial, namely that it is $A(X)$-linear. This is a straightforward computation. Hence the correct local interpretation of the curvature is as a matrix of two-forms.

30. April 6, 2018

30.1. Connections, continued.

Remark 235. Suppose we are in the case of a line bundle. Recall that we wrote, locally, $\theta = \partial h \cdot h^{-1}$. If we differentiate one more time we find that

$$0 = \partial^2 h = h\partial\theta - \theta\partial h = h(\partial\theta - \theta \wedge \theta).$$

This is the Cartan structure equation. In particular since θ is just a form in the case of line bundles, we have that $\partial\theta = 0$ whence we may write $\Theta = \bar{\partial}\theta = d\theta$.

Last time we recalled the homomorphism

$$H^1(X; \mathcal{O}_X^\times) \to H^{1,1}X \subset H^2(X; \mathbb{C})$$

sending $L \mapsto [\Theta_L]$. Recall now the long exact sequence associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

yields in particular a connecting homomorphism

$$c_1 : H^1(X; \mathcal{O}_X^\times) \to H^2(X; \mathbb{Z})$$

which we will call the **first Chern class**. The first Chern class assigns to each line bundle an integral degree 2 cohomology class. However, $H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{C})$ as well so one might ask whether it is related to the curvature.

Lemma 236. *There is a fundamental relation*

$$c_1(L) = \frac{i}{2\pi}[\Theta_L]$$

inside the image of $H^2(X;\mathbb{Z}) \cap H^{1,1}X$ *in* $H^2(X;\mathbb{C})$.

In particular we learn that the Chern class is a $(1,1)$-class. By the way, the intersection $H^2(X;\mathbb{Z}) \cap H^{1,1}X$ is called the set of integral Hodge classes. The proof of this lemma is nontrivial, so we will only sketch the proof.

Proof sketch. We first write $c_1(L)$ in Čech cohomology. Fix an open cover $\mathcal{U}$ where each U_α is simply-connected. L has transition functions $g_{\alpha\beta} \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$. We can choose $f_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ such that

$$\exp(2\pi i f_{\alpha\beta}) = g_{\alpha\beta}.$$

The $f_{\alpha\beta}$ on the triple overlap satisfy

$$c_{\alpha\beta\gamma} := f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} \in \mathbb{Z}$$

by the cocycle condition for $g_{\alpha\beta}$. The class of this two-cocycle $[(c_{\alpha\beta\gamma})_{\alpha\beta\gamma}] \in H^2(X;\mathbb{Z})$ is exactly the Chern class (we examined the connecting homomorphism last quarter).

The nontrivial part of the proof is to rewrite this class in terms of de Rham cohomology. Pick $\{\rho_\alpha\}$ to be a partition of unity subordinate to $\mathcal{U}$. We define one-forms

$$\phi_\alpha = \sum_\gamma \rho_\gamma df_{\gamma\alpha} \in A^1(U_\alpha)$$

Notice that $df_{\alpha\beta} = \phi_\alpha - \phi_\beta$ (maybe the sign here is wrong) using the fact that $c_{\alpha\beta\gamma} = 0$. Hence $d\phi_\alpha$ glues to a closed global two-form that we denote by $\omega \in A^2X$. We claim that $[\omega] = c_1(L) \in H^2(X;\mathbb{Z})$; we will not prove this.

Now fix a Hermitian metric h on L. Say L is trivialized by sections s_α on U_α satisfying $s_\alpha = \psi_{\alpha\beta} s_\beta$ for $\psi_{\alpha\beta}$ a nowhere vanishing holomorphic function. The functions defining the metric are $h_\alpha = h(s_\alpha, s_\alpha) = |\psi_{\alpha\beta}|^2 h_\beta$. Now the connection one-forms satisfy

$$\begin{aligned}\theta_\alpha - \theta_\beta &= d \log h_\alpha - \partial \log h_\beta = \partial \log |\psi_{\alpha\beta}|^2 \\ &= d\psi_{\alpha\beta}/\psi_{\alpha\beta}.\end{aligned}$$

Whoops we made a mistake here, we'll finish it next time. □

The following result is the solution to the Hodge conjecture for $(1,1)$-classes.

Theorem 237 (Lefschetz theorem on $(1,1)$-classes)**.** *If X is compact Kähler then a class $\alpha \in H^2(X;\mathbb{C})$ is an integral Hodge class, i.e. is in $H^{1,1}X \cap H^2(X;\mathbb{Z})$, if and only if α is the first Chern class of some line bundle.*

Proof. We look again at the long exact sequence,

$$H^1(X;\mathcal{O}_X^\times) \to H^2(X;\mathbb{Z}) \to H^2(X;\mathcal{O}_X)$$

The middle guy maps to $H^2(X;\mathbb{C}) = H^{2,0}X \oplus H^{1,1}X \oplus H^{0,2}X$. By the Dolbeault theorem we know that $H^{0,2}X \cong H^2(X;\mathcal{O}_X)$. It is a straightforward exercise to check that the second map in the sequence above is precisely the projection to $H^{0,2}X$, under this Dolbeault isomorphism. Exactness now shows that an integral Hodge class is a first Chern class. □

Remark 238. Let $D \subset X$ be an analytic hypersurface. Then we have constructed a line bundle $L = \mathcal{O}_X(D)$. There is a yet another class $\eta_D \in H^2(X;\mathbb{Z})$ that is Poincaré dual to $[D] \in H_{2n-2}(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$. It turns out that

$$c_1(L) = \eta_D.$$

Remark 239. Assume that ω, the Kähler form of X, is in $H^{1,1}(X;\mathbb{Q}) := H^{1,1}X \cap H^2(X;\mathbb{Q})$. This, we will see later, means that X is projective. Then the hard Lefschetz theorem says that the operator L^{n-1}, which passes to rational cohomology in this case, gives us an isomorphism

$$L^{n-1} : H^{1,1}(X;\mathbb{Q}) \to H^{n-1,n-1}(X;\mathbb{Q}).$$

Remark 240. Let X be a projective manifold. Then the million-dollar Hodge conjecture states that every class in $H^{p,p}X \cap H^{2p}(X;\mathbb{Q})$ is (the Poincaré dual of) a linear combination $\sum_i a_i [Z_i]$ where Z_i are analytic subvarieties of X.

In fact there are counterexamples if one replaces $\mathbb{Q}$ by $\mathbb{Z}$. We generally know how to produce rational Hodge classes only from geometry (divisors, etc.), hence why one might believe in this conjecture.

31. April 9, 2018

Recall that last time we were proving that $c_1(L) = [i\Theta_L/2\pi]$ where Θ_L is the curvature of any Hermitian metric on the line bundle L. We had chosen $\{U_\alpha\}$ simply connected opens covering the base, yielding $g_{\alpha\beta}$ transition functions for L. We then defined $g_{\alpha\beta} = \exp(2\pi i f_{\alpha\beta})$ and chose a partition of unity $\{\rho_\alpha\}$ associated to the cover. Then we defined

$$\phi_\alpha := \sum_\gamma \rho_\gamma df_{\gamma\alpha} \in A^1(U_\alpha).$$

It is easy to check that $df_{\alpha\beta} = \phi_\alpha - \phi_\beta$ whence the $d\phi_\alpha$ glue to a global two-form ω such that $[\omega] = c_1(L)$.

Now fix a Hermitian metric h on L. On U_α choose s_α a section corresponding to $x \mapsto (x, 1)$, where $1 \in \mathbb{C}$ under our trivialization. More precisely if we denote our trivializations by Φ_α. Then we have that $\Phi_\alpha^{-1} \circ s_\alpha = \Phi_\beta^{-1} \circ s_\beta$. This precisely means that

$$s_\beta = g_{\alpha\beta} s_\alpha.$$

Notice that this is not the gluing condition! Next note that $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$. It follows that for the Chern form we have

$$\theta_\beta - \theta_\alpha = \partial \log |g_{\alpha\beta}|^2 = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} = 2\pi i df_{\alpha\beta}.$$

Using this computation we can write

$$\begin{aligned}\phi_\alpha &= \frac{i}{2\pi} \sum_\gamma \rho_\gamma (\theta_\alpha - \theta_\gamma) \\ &= \frac{i}{2\pi} \theta_\alpha - \Psi,\end{aligned}$$

where we write $\Psi = \frac{i}{2\pi} \sum_\gamma \rho_\gamma \theta_\gamma$, a global one-form. Since the Chern class is given as $d\phi_\alpha$ on U_α, taking exterior derivatives we find that

$$d\phi_\alpha = \frac{i}{2\pi} d\theta_\alpha - d\Psi = \frac{i}{2\pi} \Theta_L - d\Psi.$$

Here we implicitly use that $\partial \Theta_L = 0$ which we showed last time. Since $d\Psi$ is a global exact form. This concludes the proof.

Example 241. Let $X = \mathbb{P}^n$ and the line bundle $L = \mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ over X. Recall that as a set L is defined as the set of pairs (ℓ, p) such that $p \in \ell$. Notice that L acquires a Hermitian metric by restriction. Consider the open set $U_0 = \{z_0 \neq 0\}$. Choose over U_0 the section

$$s_0([1 : z_1 : \ldots : z_n]) = (1, z_1, \ldots, z_n)$$

and write

$$h_0 = h(s_0, s_0) = 1 + |z_1|^2 + \cdots + |z_n|^2.$$

The curvature of the Chern connection on L is now

$$\frac{i}{2\pi} \Theta_0 = -\frac{i}{2\pi} \partial\bar{\partial} \log(1 + |z_1|^2 + \cdots + |z_n|^2) = -\omega_{\mathsf{FS}}.$$

We conclude that $c_1(\mathcal{O}_{\mathbb{P}^n}(-1)) = [\omega_{\mathsf{FS}}] \in H^{1,1}(\mathbb{P}^n)$.

Example 242. Let (X, h) be a complex manifold equipped with a Hermitian metric, i.e. h is a Hermitian metric on $T'X$ given locally by

$$h_{jk} = h(\partial/\partial z_j, \partial/\partial z_k), \qquad H = (h_{jk}).$$

Dually we have the cotangent bundle $T^{*'}X$, which inherits the inverse metric $h^{jk} = h(dz_j, dz_k)$. The canonical bundle is the top exterior power of this bundle, $\omega_X = \Lambda^n \Omega^1_X$. It in turn inherits a metric which locally we can write as follows:

$$h(dz_1 \wedge \cdots \wedge dz_n, dz_1 \wedge \cdots \wedge dz_n) = \det H^{-1} = 1/\det H.$$

The curvature is computed

$$\Theta_{\omega_X} = \partial\bar{\partial} \log \det H.$$

The sign of this metric is crucial in classification of manifolds.

31.1. **Harmonic forms.** We now turn to harmonic theory for forms valued in line bundles. Let L be a holomorphic line bundle on a compact complex manifold X. We always have an operator

$$\bar{\partial} : A^{p,q}(X; L) \to A^{p,q+1}(X; L)$$

satisfying $\bar{\partial}^2 = 0$. This means we can defined new cohomology groups.

Definition 243. The **Dolbeault cohomology groups of** L are

$$H^{p,q}(X; L) := \frac{\ker\left(\bar{\partial} : A^{p,q}(X; L) \to A^{p,q+1}(X; L)\right)}{\operatorname{im}\left(\bar{\partial} : A^{p,q-1}(X; L) \to A^{p,q}(X; L)\right)}.$$

Of course, we have that $H^{p,q}X = H^{p,q}(X; \mathcal{O}_X)$. More generally we have a Dolbeault complex of L:

$$0 \to \mathcal{A}^{p,0}_X(L) \to \mathcal{A}^{p,1}_X(L) \to \cdots \to \mathcal{A}^{p,n}_X(L) \to 0.$$

This is just what we had before, tensored with the sheaf L. Hence we retain all the properties from before.

Proposition 244. *The Dolbeault complex satisfies the following properties.*

(1) it is a resolution of $\Omega^p_X \otimes L$*;*
(2) the $\mathcal{A}^{p,q}_X(L)$ *are fine sheaves.*

By the fact that fine sheaves compute sheaf cohomology we conclude the corresponding Dolbeault theorem.

Theorem 245 (Dolbeault)**.** *There is a natural isomorphism*

$$H^q(X; \Omega^p_X \otimes L) \cong H^{p,q}(X; L).$$

Now fix h a Hermitian metric on X and h_L a Hermitian metric on L. We obtain a Hermitian inner product on the space $\mathcal{A}^{p,q}(X; L)$ by

$$\langle \alpha_1 \otimes s_1, \alpha_2 \otimes s_2 \rangle_L = \int_X h(\alpha_1, \alpha_2) h_L(s_1, s_2) d\mathrm{vol}(g).$$

We can use this inner product to define $\bar{\partial}^*$, the adjoint of $\bar{\partial}$. This leads to the Laplacian

$$\bar{\Box} = \bar{\partial} \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial},$$

a second-order elliptic operator. This allows us to define the $\bar{\partial}$-harmonic forms with coefficients in L:

$$\mathcal{H}^{p,q}(X; L) := \ker \bar{\Box}.$$

As before we have a Hodge-type result.

Theorem 246 (Hodge)**.** *The inner product data yields isomorphisms*

$$\mathcal{H}^{p,q}(X;L) \cong H^{p,q}(X;L).$$

We are interested, in particular, in the Dolbeault cohomology of "positive" line bundles. We will show that for positive line bundles $H^{p,q}(X;L) = 0$ for all $p+q > n = \dim X$

Definition 247. Let X be a compact complex manifold. We say that a line bundle $L \to X$ is **postiive** if its first Chern class can be represented by a (closed, $(1,1)$) form ω with a positive-definite associated Hermitian form.

Recall that this Hermitian form is defined as follows. We write

$$\omega = \frac{i}{2\pi} \sum_{j,k} f_{jk} dz_j \wedge d\bar{z}_k.$$

The positivity condition requires that $(f_{jk})^{j,k}$ is a positive-definite Hermitian matrix, i.e. $f = \overline{{}^t f}$ and $\overline{{}^t v} f v > 0$ for each $v \neq 0$. Notice that this implies automatically that X is Kähler! However, we will find that is shows much more, i.e. that X is projective.

32. April 11, 2018

Let X be a K3 surface. It turns out that this implies $\omega_X \cong \mathcal{O}_X$. We have $H^1(X;\mathcal{O}_X) = 0$ and $H^2(X;\mathcal{O}_X) = \mathbb{C}$.

There is another important class of surfaces known as Enriques surfaces, Y smooth projective such that there is a smooth finite map $X \to Y$ and $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$ but $\omega_Y \neq \mathcal{O}_Y$. On the Enriques surface however $H^1(Y;\mathcal{O}_Y) = 0$ and $H^2(Y;\mathcal{O}_Y) = 0$. If we now look at the exponential sequence for Y we find that

$$H^1(Y;\mathcal{O}_Y) \to H^1(Y;\mathcal{O}_Y^\times) \to H^2(X;\mathbb{Z}) \to H^2(Y;\mathcal{O}_Y)$$

whence the Chern class c_1 is an isomorphism. In other words, the first Chern class identifies the isomorphism classes of line bundles. In particular we have $2 \cdot c_1(\omega_Y) = 0$ but $c_1(\omega_Y) \neq 0$ so we find a torsion class.[2] Confused about this

32.1. A few words on the Hodge conjecture.

Let $Z^{n-k} \subset X^n$ be a closed submanifold of codimension K. We have a homology class $[Z] \in H_{2n-2k}(X;\mathbb{Z})$. This group is Poincaré dual to $H^{2k}(X;\mathbb{Z})$ and we write the corresponding class, after passing to complex coefficients, as η_Z. With complex coefficients we have a pairing

$$H^{2k}(X;\mathbb{C}) \to H^{2n-2k}(X;\mathbb{C})^*$$

This pairing is given by

$$\int_Z \omega|_Z = \int_X \alpha \wedge \omega$$

where α is any form representing η_Z. We claim that $\eta_Z \in H^{2k}(X;\mathbb{Z}) \cap H^{k,k}X$. To see this, note that for each ω we have

$$\begin{aligned} \int_X \alpha \wedge \omega = \int_Z \omega|_Z = \int_Z \omega^{n-k,n-k}|_Z = \int_X \alpha \wedge \omega^{n-k,n-k} \\ = \int_X \alpha^{k,k} \wedge \omega^{n-k,n-k} = \int_X \alpha^{k,k} \wedge \omega. \end{aligned}$$

This implies that $\alpha = \alpha^{k,k}$. The Hodge conjecture states that every Hodge class (over $\mathbb{Q}$) comes from geometry in this way (one should note that it is a conjecture in algebraic geometry – it is known to be false for Kähler manifolds).

32.2. Positive line bundles.

Recall that L a line bundle on X a compact complex manifold is positive if $c_1(L)$ can be represented by a closed $(1,1)$-form ω which is positive (i.e. its associated Hermitian matrix is positive definite). We remarked last time that if $L \to X$ is positive then any representative of $c_1(L)$ yields a Kähler structure on X.

Example 248. Let $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$. We saw that $c_1(L) = [\omega_{FS}]$ whence L is positive. Philosophically speaking, according to Kodaira embedding, this is the only important positive line bundle.

Our first goal is the following.

Theorem 249 (Kodaira-Akizuki-Nakano). *If L is positive then $H^{p,q}(X;L) = H^q(X;\Omega_X^p \otimes L) = 0$ for all $p+q > n = \dim X$.*

[2] For more details on such computations, see for instance, Mihnea's notes: http://www.math.northwestern.edu/~mpopa/483-3/notes/notes.pdf.

Proposition 250 ($\partial\bar\partial$ Lemma, or the principle of two types)**.** *Let X be a compact Kähler manifold and $\alpha \in A^kX$ such that $\partial\alpha = \bar\partial\alpha = 0$. If α is also ∂-exact or $\bar\partial$-exact then in fact $\alpha = \partial\bar\partial\beta$ for $\beta \in A^{k-2}X$.*

The proof is not complicated but it uses the full force of Hodge theory.

Proof. Say that $\alpha = \bar\partial\gamma$ for some γ. Since X is Kähler we have our decomposition $A^kX = \mathcal{H}^kX \oplus \operatorname{im}\Delta$ and we know that $\Delta = 2\bar\Box = 2\Box$. Decompose $\gamma = \eta + \Delta\nu$ where $\Delta\eta = 0$. It follows that $\bar\Box\eta = 0$ whence $\bar\partial\eta = 0$. Now $\alpha = \bar\partial\gamma = \bar\partial\eta + 2\bar\partial\Box\nu = 2\bar\partial(\partial\partial^* + \partial^*\partial)\nu$. Since α is closed, applying ∂ gives us

$$\begin{aligned} 0 = \partial\alpha &= -2\partial\partial\bar\partial\partial^*\nu + 2\partial\bar\partial\partial^*\partial\nu \\ &= 2\partial\bar\partial\partial^*\partial\nu \\ &= -2\partial\partial^*\bar\partial\partial\nu \end{aligned}$$

where we have used $\bar\partial\partial^* = -\partial^*\bar\partial$ (see earlier in notes). This shows that $\partial^*\bar\partial\partial\nu \in \ker\partial \cap \operatorname{im}\partial^*$. Since ∂ and ∂^* are adjoint operators, this intersection is zero. Hence we have

$$\alpha = -2\partial\bar\partial\partial^*\nu,$$

as desired. □

Lemma 251. *Let L be a line bundle on a compact Kähler manifold with $c_1(L) = [\omega]$ for ω a closed $(1,1)$-form. Then there exists a Hermitian metric h_L on L such that*

$$\omega = \frac{i}{2\pi}\Theta_L.$$

Proof. Let h_0 be an arbitrary Hermitian metric on L. We obtain from the associated Chern connection the curvature form $\Theta_0 \in A^{1,1}X$. In de Rham cohomology we have that $c_1(L) = [\frac{i}{2\pi}\Theta_0] = [\omega]$. In other words,

$$\eta := \omega - \frac{i}{2\pi}\Theta_0$$

is $\bar\partial$-exact. But of course ω and Θ_0 are both ∂- and $\bar\partial$-closed. Hence $\partial\eta = \bar\partial\eta = 0$. Now the $\partial\bar\partial$-lemma above implies that $\eta = i/2\pi \cdot \psi$ for some $\psi \in A^0X$. Hence

$$\begin{aligned} \omega &= \frac{i}{2\pi}\Theta_0 + \frac{i}{2\pi}\partial\bar\partial\psi \\ &= \frac{i}{2\pi}\partial\bar\partial\log h_0 + \frac{i}{2\pi}\partial\bar\partial\log e^{\psi} \\ &= -\frac{i}{2\pi}\partial\bar\partial\log(h_0e^{-\psi}). \end{aligned}$$

We conclude that the desired metric is $h_L = h_0e^{-\psi}$. □

Remark 252. Fix a metric h_L on $L \to X$ and write ∇ for the associated Chern connection. We decomposed

$$\nabla = \nabla' + \nabla''$$

where recall that $\nabla'' = \bar\partial$. Let us for simplicity write $\nabla' = \partial$ which will hopefully not be cause for confusion. These of course can be extended to any (p,q)-forms:

$$\begin{aligned} \bar\partial(\alpha \otimes s) &= \bar\partial\alpha \otimes s \\ \partial(\alpha \otimes s) &= \partial\alpha \otimes s + (\theta_L \wedge \alpha) \otimes s, \end{aligned}$$

where locally s is a trivialization for L and $\nabla s = \theta_L \otimes s$.

Lemma 253. *We have that* $\partial\bar{\partial} + \bar{\partial}\partial = \Theta_L$ *as operators on* $A^{*,*}(X; L)$.

Proof. Simply notice that $\Theta_L = \nabla^2 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial\bar{\partial} + \bar{\partial}\partial$. □

Of course here we view the curvature as the operator given by wedging with the curvature two-form.

The next step will be the generalized Kähler identities, which will just be the generalization of the Kähler identities for the case of line bundles.

33. April 13, 2018

Recall that we have an inner product on $A^{p,q}(X;L)$ coming from the Hermitian metrics on X and L. If $\partial = \nabla'$ and $\bar\partial = \nabla''$ are the pieces of the Chern connection using the notation from last time, we consider their adjoints with respect to this inner product.

For $\alpha, \beta \in A^{p,q}(U)$ where β has compact support and $s \in \Gamma(U;L)$ a local trivializing section, we compute

$$\begin{aligned}
\partial^*(\alpha\otimes s), \beta\otimes s)_L &= (\alpha\otimes s, \partial(\beta\otimes s))_L \\
&= (\alpha\otimes s, \partial\beta\otimes s + (\theta\wedge\beta)\otimes s)_L \\
&= \int_X (\alpha, \partial\beta + \theta\wedge\beta)\cdot(s,s)_L \mathrm{vol}(g) \\
&= \int_X (\alpha, f d\beta + \partial f\wedge\beta)\mathrm{vol}(g) \\
&= \int_X (\alpha, \partial(f\beta))\mathrm{vol}(g) \\
&= (\alpha, \partial(f\beta)) = (\partial^*\alpha, f\beta) \\
&= \int_X (\partial^*\alpha, \beta)(s,s)\mathrm{vol}(g) \\
&= (\partial^*\alpha\otimes s, \beta\otimes s)_L.
\end{aligned}$$

Here we have written $f :== h_L(s,s)$ and used that $\theta = \partial \log f.$. We conclude that

$$\partial^*(\alpha\otimes s) = \partial^*\otimes s.$$

Now let us consider the special case in which L is positive. In this case there exists ω a closed positive $(1,1)$-form with $c_1(L) = [\omega]$. Recall all the machinery we introduced in the Kähler case:

- we have a Lefschetz operator $L(\alpha\otimes s) = L\alpha\otimes s$,
- its adjoint $\Lambda(\alpha\otimes s) = \Lambda\alpha\otimes s$,
- generalized Kähler identities; on the space $A^{p,q}(X;L)$ we have

$$\partial^*\bar\partial^* + \bar\partial^*\partial^* = 2\pi i\Lambda \qquad [\Lambda, \bar\partial] = -i\partial^*.$$

Proof of the generalized Kähler identities. For the first identity we use the lemma from last time: there exists a metric h_L such that $\omega = \frac{i}{2\pi}\Theta_L$. As operators we have that $\Theta_L = -2\pi i L$ (this L being the Lefschetz operator, not the line bundle!). We also know that $\Theta_L = \partial\bar\partial + \bar\partial\partial$. Now pass to adjoints (the sign is coming from the conjugation in the adjoint) and we are done.

For the second identity the remarks above show us that the adjoint operators just act in the differential form component whence the commutator follows immediately from the usual Kähler identity. □

Recall that we wish to prove the following statement, one of the most fundamental results in algebraic geometry.

Theorem 254 (Kodaira-Akizuki-Nakano vanishing theorem). *If $L \to X$ is positive then $H^{p,q}(X;L) \cong H^q(X;\Omega^p_X\otimes L) = 0$ for all $p+q > n$.*

Proof. We write $c_1(L) = [\omega]$ where ω is positive. The proof is an exercise in the identities above. First notice that by Hodge theory we have

$$H^{p,q}(X;L) \cong \mathcal{H}^{p,q}(X;L).$$

It suffices to show that $\mathcal{H}^{p,q}(X;L) = 0$ for $p+q > n$.

Take any element $\alpha \in \mathcal{H}^{p,q}(X;L)$. In particular

$$\bar\partial\alpha = 0 \qquad \bar\partial^*\alpha = 0.$$

We now compute using the Kähler identities

$$\begin{aligned}
\|\Lambda\alpha\|_L^2 &= (\Lambda\alpha, \Lambda\alpha)_L \\
&= \frac{i}{2\pi}(\Lambda\alpha, (\partial^*\bar\partial^* + \bar\partial^*\partial^*)\alpha)_L \\
&= \frac{i}{2\pi}(\Lambda\alpha, \bar\partial^*\partial^*\alpha)_L \\
&= \frac{i}{2\pi}(\Lambda\alpha, \bar\partial\Lambda\alpha, \partial^*\alpha)_L \\
&= \frac{i}{2\pi}([\bar\partial, \Lambda]\alpha, \bar\partial\Lambda\alpha, \partial^*\alpha)_L \\
&= -\frac{1}{2\pi}(\partial^*\alpha, \partial^*\alpha)_L \\
&= -\frac{1}{2\pi}\|\partial^*\alpha\|^2.
\end{aligned}$$

This implies that $\partial^*\alpha = \Lambda\alpha = 0$, i.e. α is primitive. But by a result from last quarter since $\deg\alpha = p+q < n$ we must have $\alpha = 0$ (the same result applies for forms with values in a line bundle). □

Remark 255. What is often called “Kodaira vanishing” is the case where $p = n$:

$$H^q(X; \omega_X \otimes L) = 0$$

for all $q > 0$ and all positive line bundles $L \to X$.

Remark 256. Take E any Hermitian vector bundle. There is no naive analog of the Kodaira-Akizuki-Nakano vanishing theorem. There is something we can do, however: there is a Hodge star operator

$$*_E : A^{p,q}(X;E) \to A^{n-p,n-q}(X, E^\vee)^\vee$$

which induces isomorphism when passing to harmonic forms (or cohomology):

$$\mathcal{H}^{p,q}(X;E) \xrightarrow{\sim} \mathcal{H}^{n-p,n-q}(X;E^\vee)^\vee.$$

The resulting duality theorem we obtain is known as **Serre duality**:

$$H^q(X;\Omega_X^p \otimes E) \cong H^{n-q}(X;\Omega_X^{n-p} \otimes E^\vee)^\vee.$$

In particular, for $p = 0$, we have the famous relation

$$H^q(X;E) \cong H^{n-q}(X;\omega_X \otimes E^\vee)^\vee.$$

You can find the details in, for instance, Huybrecht's book.

If we apply Serre duality to Kodaira-Akizuki-Nakano vanishing we find the following. If L is a positive line bundle then

$$H^q(X;\Omega_X^p \otimes L^{-1}) = 0$$

for $p+q<n$. In particular, for $p=0$ we have the useful relation that

$$H^q(X,L^{-1})=0$$

for all $q<n$.

We can finally compute the cohomology of all the line bundles on projective space.

Example 257 (Projective space)**.** Let $X=\mathbb{P}^n$. We will use the fact that $\omega_{\mathbb{P}^n}\cong\mathcal{O}_{\mathbb{P}^n}(-n-1)$ (we'll see this when we do lots of examples soon, probably next class). We know from the exponential sequence that all line bundles are written as $\mathcal{O}_{\mathbb{P}^n}(k)$ for $k\in\mathbb{Z}$.

The first statement we make is that

$$H^i(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(k))=0 \qquad 0<i<n \text{ for all } k.$$

This is a consequence of Kodaira vanishing. If $k<0$ we apply the result in the remark above. If $k\geq 0$ we apply Serre duality:

$$H^i(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(k)\cong H^{n-i}(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(-n-1-k))^\vee$$

and again apply the remark above.

The next statement we make is that by Serre duality

$$H^n(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(k))\cong H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(-n-1-k))^\vee.$$

In other words, it suffices to understand the zeroth cohomology.

The final statement we make is that

$$H^0(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(k))=\begin{cases}0 & k<0\\ \text{monom. of deg } k \text{ in } n+1 \text{ vars} & k\geq 0\end{cases}$$

In particular the dimension of this space of monomials is $N_k=\binom{n+k}{k}$.

Next time we'll take a break and return to geometry and discuss divisors, etc.

34. April 16, 2018

We'd like to use Kodaira vanishing to prove the Lefschetz hyperplane theorem. Before we do this we'll review some very basic facts about hypersurfaces and submanifolds.

34.1. Generalities on submanifolds.

Lemma 258. *If $Y \subset X$ is a submanifold of a complex manifold then there exists a natural inclusion $T_Y \hookrightarrow T_X|_Y$ (in fact a monomorphism of of vector bundles). In particular there exists a short exact sequence of vector bundles*

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0$$

where $N_{Y/X}$ is the normal bundle of Y in X. It is a vector bundle of rank $\dim X - \dim Y$, the codimension of Y.

It is important to note that this short exact sequences is not split, unlike the smooth case!

Proof. Let (U_i, ϕ_i) be an atlas for X such that

$$\phi_i(U_i \cap Y) = \phi_i(U_i) \cap \{z_{m+1} = \cdots = z_n = 0\}$$

where $m = \dim Y, n = \dim X$. Next let g_{ij} be the transition functions of T_X. By construction

$$g_{ij} = \mathcal{J}(\phi_{ij}) \circ \phi_j.$$

Restricting to Y under the coordinates we've chosen, we obtain a matrix

$$g_{ij}|_Y \begin{pmatrix} g_{ij}^T & * \\ 0 & h_{ij} \end{pmatrix}$$

where g_{ij}^Y are the transition functions for T_Y (from the induced atlas for Y). We leave the following general fact as an exercise: that the transition function is block of this form induces an inclusion of vector bundles $0 \to T_Y \to T_X|_Y$ and h_{ij} are the transition functions of the cokernel. □

Corollary 259 (Adjunction formula)**.** *With the same assumptions as above, we have*

$$\omega_Y \cong \omega_X|_Y \otimes \det N_{Y/X}. \tag{1}$$

For some reason the word adjunction in algebraic geometry sometimes mean passing from an ambient space to a subspace.

Proof. Take the top exterior power of the short exact sequence in the lemma, then pass to duals (the cotangent bundles). □

Lemma 260. *The canonical bundle of projective space is given*

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Proof. The only work to be done is to determine where the $-n-1$ argument comes from. Fix the standard open cover of $\mathbb{P}^n$ where the $U_i = \{z_i \neq 0\}$ and our charts send $(z_0 : \cdots : z_n) \mapsto (w_1 = z_0/z_i, \ldots, \hat{i}, \ldots, w_n = z_n/z_i)$. Then the transition functions are written

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} = (j+1, i) \circ \tilde{\phi}_{ij}$$

where $(j+1, i)$ is a permutation of sign $i-j-1$ and

$$\tilde{\phi}_{ij} : (w_1, \ldots, w_n) \mapsto (w_i^{-1}w_1, \ldots, w_i^{-1}w_{i-1}, w_i^{-1}, w_i^{-1}w_{i+1}, \ldots, w_i^{-1}w_n).$$

The canonical bundle is $\omega_{\mathbb{P}^n} = \det(T_{\mathbb{P}^n}^\vee)$ so it suffices to show that $\mathcal{O}_{|\mathbb{P}^n}(n+1)$ is given by the transition functions $\det(\mathcal{J}(\phi_{ij} \circ \phi_j))$.

We compute

$$\det \mathcal{J}(\phi_{ij}) = (-1)^{i-j-1} \det \left(\frac{\partial \tilde{\phi}_{ij}^k}{\partial w_l} \right)_{kl} = (-1)^{i-j} \frac{1}{w_i^{n+1}}.$$

After composing with ϕ_j we obtain

$$(-1)^{i-j} \left(\frac{z_j}{z_i} \right)^{n+1}$$

and we are done. □

Remark 261. Here is another proof: we have the Euler sequence, a short exact sequence of vector bundles:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

This sequence is just coming (after tensoring) from the construction of the tautological line bundle $0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \oplus_{n+1}\mathcal{O}_{\mathbb{P}^n}$. The tensoring preserves the inclusion as we are tensoring by a line bundle. Next we pass to determinants to obtain the result of the lemma above.

Corollary 262. *For all $k > 0$ we have $H^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{\otimes k}) = 0$.*

Proof. Follows immediately from our discussion last time that negative line bundles have no sections. □

Another way of stating this result is to say that the Kodaira dimension $K(\mathbb{P}^n) = -\infty$.

Now let's try to understand the normal bundle so that the adjunction formula is useful.

Proposition 263. *If $Y \subset X$ is a smooth hypersurface (i.e. a complex submanifold of codimension one) let L be the line bundle $\mathcal{O}_X(Y)$, the line bundle associated to the hypersurface. Then*

$$N_{Y/X} \cong L|_Y = \mathcal{O}_X(Y)|_Y =: \mathcal{O}_Y(Y)$$

and

$$\omega_Y \cong \omega_X|_Y \otimes \mathcal{O}_Y(Y) =: \omega_X(Y)|_Y.$$

Proof. The second isomorphism follows immediately from applying the adjunction formula to the first isomorphism.

The first isomorphism can be done through a local approach using transition functions. We will think about it slightly differently. Suppose Y is locally on U_i given by $s_i = 0$ where $s_i \in \mathcal{O}_X$. We can write $s_i = g_{ij}s_j$ with g_{ij} the transition functions for L (this was how we constructed L!). Let's now think about what the forms ds_i are. Dualize the sequence defining the normal bundle and notice that

$$\begin{array}{ccccc} 0 \to N_{Y/X}^\vee \to \Omega_X^1|_Y & & \to \Omega_Y^1 \to 0 \\ ds_i|_Y & & \mapsto 0. \end{array}$$

Hence we think of ds_i as a local section of $N_{Y/X}^\vee$, trivializing it. Recall we have that $ds_i = h_{ji} ds_j$ but differentiating the gluing condition for s_i yields

$$ds_i = dg_{ij} s_j + g_{ij} s_j.$$

If now consider $U_i \cap U_j \cap Y$ the first term vanishes (exercise!). We conclude that $g_{ij} = h_{ji}$ whence $N_{Y/X}^\vee$ is isomorphic to $L^\vee$ and we are done. □

Let's go back to $\omega_Y \cong \omega_X|_Y \otimes \mathcal{O}_Y(Y)$. There is a surjective residue map

$$\omega_X \otimes \mathcal{O}_X(Y) \xrightarrow{\mathsf{Res}} \omega_Y \to 0.$$

Locally, choose coordinates such that $z_n = 0$. Then a local section of the left is written $f dz_1 \wedge \cdots \wedge dz_n / z_n$. We map this to the section $f|_Y dz_1 \wedge \cdots \wedge dz_{n-1}$. This is clearly the complex analytic notion of residue that we are familiar with. The kernel of the residue map is those where $f|_Y \equiv 0$ i.e. $f = z_n g$ for some g. Hence the kernel is exactly ω_X. In fact, this is true for every p:

$$0 \to \Omega_X^p \to \Omega_X^p(\log Y) \xrightarrow{\mathsf{Res}} \Omega_Y^{p-1} \to 0,$$

the famous residue sequence for forms with poles along a divisor.

35. April 18, 2018

Remark 264. Given any submanifold $Z \subset X$ (of arbitrary codimension), we can consider the ideal

$$\mathcal{I}_Z(U) = \{f \in \mathcal{O}_X(U) \mid f|_Z \equiv 0\}.$$

Consider the quotient $\mathcal{I}_Z/\mathcal{I}_Z^2 = \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_Z$, which we can also write as $\mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. Indeed, one can think of $\mathcal{I}_Z/\mathcal{I}_Z^2$ as a vector bundle on Z of rank codimension of Z. Moreover $N_{Z/X} \cong (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$. In other words the normal bundle can be constructed from the ideal sheaf.

Definition 265. A **complete intersection** in a complex manifold X is a submanifold Z of the form $Z = D_1 \cap \cdots D_r$ where $r = \operatorname{codim}_X Z$ and D_i are hypersurfaces in X.

The restriction of Z being a manifold is not necessary, we could equally well have it be singular. This is a very special condition because something of codimension k need not be cut out by exactly k equations! In other words the result from algebra that we know (Krull's hauptidealsatz) does not globalize! We can think of this as an iterative sequence of hypersurfaces (see the following exercise).

The following result is a very important calculational tool.

Exercise 266. If $Z \subset \mathbb{P}^n$ is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_r$ then $\omega_Z \cong \mathcal{O}_{\mathbb{P}^n}(d_1 + \cdots + d_r - n - 1)$.

Example 267. Recall that we saw that $S \subset \mathbb{P}^3$ a smooth hypersurface of degree 4, so the exercise above immediately implies that

$$\omega_S \cong \mathcal{O}_S.$$

Indeed, in a moment we will also see that $H^1(S; \mathbb{C}) = 0$. We call this a K3 surface.

Example 268. Let $S \subset \mathbb{P}^4$ be a complete intersection of type (2,3). THen

$$\omega_S \cong \mathcal{O}_S.$$

A small challenge for you: can you write down all the possible ways that a K3 surface can be written as a complete intersection in projective space.

Theorem 269 (Weak Lefschetz theorem, Lefschetz hyperplane theorem)**.** *If $D \subset X$ is a hypersurface in a compact Kähler manifold of dimension n such that $\mathcal{O}_X(D)$ is positive, then the restriction map*

$$\mathsf{restr} : H^i(X;\mathbb{C}) \to H^i(D;\mathbb{C})$$

is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

The main example of course is when $D = X \cap H$ where $X \subset \mathbb{P}^n$ and H is a hyperplane in $\mathbb{P}^n$. Notice moreover that this can be iterated to obtain an analogous result for complete intersections.

Remark 270. This is a purely topological statement that we will prove using purely holomorphic methods. In fact, the theorem is true integrally! See chapter one of my note's on p-adic and motivic integration, where we prove this theorem over $\mathbb{Z}$ using Morse theory: http://www.math.northwestern.edu/~mpopa/571/chapter1.pdf.

Proof of the weak Lefschetz theorem. Notice that restr is a map of Hodge structures as harmonicity of forms, etc is preserved. More explicitly

$$\begin{array}{ccccc} H^i(X;\mathbb{C}) & \xrightarrow{\sim} & \oplus_{p+q=i} H^{p,q}X & \xrightarrow{\sim} & \oplus H^q(X;\Omega^p_X) \\ \downarrow & & \downarrow{\scriptstyle \oplus r_{p,q}} & & \downarrow{\scriptstyle \oplus r_{p,q}} \\ H^i(D;\mathbb{C}) & \xrightarrow{\sim} & \oplus_{p+q=i} H^{p,q}D & \xrightarrow{\sim} & \oplus H^q(D;\Omega^p_D) \end{array}$$

Hence it is enough to show that $r_{p,q} : H^q(X;\Omega^p_X) \to H^q(X;\Omega^p_D$ is an isomorphism for $p+q \leq n-2$ and injective for $p+q = n-1$.

We have a sequence of maps (one should be careful about inserting i_* in certain places below)

$$\Omega^p_X \xrightarrow{r} \Omega^p_X|_D \xrightarrow{i} \Omega^p_D.$$

Consider the defining short exact sequence of D:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

which we tensor with the locally free sheaf Ω^p_X to obtain

$$0 \to \Omega^p_X \otimes \mathcal{O}_X(-D) \to \Omega^p_X \to \Omega^p_X|_D \to 0. \tag{2}$$

Moreover we have short exact sequence for the conormal bundle

$$0 \to N^\vee_{D/X} \to \Omega^1_X|_D \to \Omega^1_D \to 0$$

but recall that $N^\vee_{D/X} \cong \mathcal{O}_D(-D)$.

Exercise 271. Given a short exact sequence

$$0 \to L \to E \to F \to 0$$

where L, E, F are vector bundles with $\operatorname{rk} L = 1$, then there is a short exact sequence

$$0 \to L \otimes \Lambda^{p-1} F \to \Lambda^p E \to \Lambda^p F \to 0.$$

The general result, for when L is not required to be a line bundle, can be found in Hartshorne exercise II.5.16.

Applying this exercise we find that

$$0 \to \Omega_D^{p-1} \otimes \mathcal{O}_D(-D) \to \Omega_X^p|_D \to \Omega_D^p \to 0. \tag{3}$$

Let us pass to the cohomology of the sequence 2:

$$H^q(X;\Omega_X^p \otimes \mathcal{O}_X(-D)) \to H^q(X;\Omega_X^p) \xrightarrow{r} H^q(X;\Omega_X^p|_D) \to H^{q+1}(X;\Omega_X^p \otimes \mathcal{O}_X(-D))$$

but from KAN vanishing we know that $H^q(X;\Omega_X^p \otimes \mathcal{O}(-D)) = 0$ for all $p+q < n$. In other words r is an isomorphism for $p+q \leq n-2$ and injective for $p+q = n-1$.

Passing to cohomology of the sequence 3 we have:

$$H^q(D;\Omega_D^{p-1} \otimes \mathcal{O}_D(-D)) \to H^q(D;\Omega_X^p|_D) \to H^q(D;\Omega_D^p) \to H^{q+1}(D;\Omega_D^{p-1} \otimes \mathcal{O}_D(-D))$$

and we can again apply KAN vanishing. It holds when $i+j < n-1$ where i, j are the indices appearing in this sequence. But due to the shift by one in indices we obtain the same result for i; i is an isomorphism for $p+q \leq n-2$ and injective for $p+q = n-1$. Thus the same holds for the composition, which is the map restr. □

This is a general principle in algebraic geometry: topological data plus vanishing results yield Hodge theoretic data or conversely, vanishing results plus Hodge theoretic data yields topological data.

Remark 272. In the Hodge theory of noncompact manifolds (due to Deligne) one works with forms with certain singularities along the boundary in a compactification.

Let $X \subset \mathbb{P}^n$. Then X is Kähler so $\mathcal{O}_{\mathbb{P}^n}(1)|_X$ is a positive line bundle. We will now prove the converse.

Theorem 273 (Kodaira's embedding theorem)**.** *If X is a compact Kähler manifold equipped with a positive line bundle then there exists an embedding $X \hookrightarrow \mathbb{P}^n$.*

One might say that this implies that X projective, though really one needs to prove one more result, that the equations defining X are algebraic.

Theorem 274 (Chow's theorem)**.** *Any compact complex submanifold of $\mathbb{P}^n$ is a projective algebraic variety.*

36. April 20, 2018

36.1. **Towards Kodaira embedding.** Since we are interested in embedding manifolds in projective space let us study maps to $\mathbb{P}^n$ in some generality. Let $f : X \to \mathbb{P}^n$ be a map of complex manifolds. Recall that on $\mathbb{P}^n$ we have a positive line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ with transition functions $g_{ij} = z_j/z_i$ on the usual charts/trivializations. Thus the pullback $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is positive and trivialized over $f^{-1}(U_i)$ with transition functions $z_j/z_i \circ f$. We obtain a map of global sections

$$f^* : H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X; L).$$

Definition 275. We say that f is **nondegenerate** if $f(X)$ not contained in any hyperplane in $\mathbb{P}^n$.

Notice that if f is nondegenerate it immediately follows that f^* is injective: a section goes to zero if and only if the map sends the section into a hyperplane on which it vanishes. So f induces a pair (L, V) on X where $V \subset H^0(X; L)$ is the image of f^*.

Conversely, we might want to construct maps to $\mathbb{P}^n$ from sections of line bundles.

Definition 276. Let L be a line bundle on X and let $V \subset H^0(X; L)$ be a linear subspace. We say that V is **basepoint-free** if for every $x \in X$ there exists a section $s \in V$ such that $s(x) \neq 0$. A choice of a subspace $V \subset H^0(X; L)$ is known as a **linear system**.

Remark 277. We might think of L as a sheaf. Then there is always a map $\Gamma(X; L) \to L_x$ where L_x is the stalk of L at x. The stalk is a rank one free $\mathcal{O}_{X,x}$-module. Consider the composition

$$\Gamma(X; L) \xrightarrow{\mathrm{ev}_x} L_x \to L_x/\mathfrak{m}_x L_x =: L(x) \cong \mathbb{C}$$

This resulting complex number is exactly the value we get if we think of vector bundles and take the fiber coordinate of the section.

Now fix a basis $s_0, \ldots, s_n$ of $V \subset H^0(X; L)$. Then there exists a map

$$f : X \to \mathbb{P}^n \qquad x \mapsto (s_0(x) : \cdots : s_n(x))$$

that is not everywhere defined. It is defined everywhere if and only if V is basepoint-free. The locus where it is not defined is known as the basepoint locus. One might object that we have made a choice of basis – however the resulting map for just the data of V is well-defined up to the action of $PGL_{n+1}(\mathbb{C})$.

Remark 278. There is a more invariant description (known as the Grothendieck convention or notation). Fix the vector space V and consider the projective space $\mathbb{P}(V)$, the space of hyperplanes in V (or the space of lines in V^*). In this notation we can define invariantly

$$X \to \mathbb{P}(V) \qquad x \mapsto \{s \in V \mid s(x) = 0\}.$$

This of course only makes sense when the right hand side is a hyperplane, which happens exactly when V is basepoint-free.

From these definitions we conclude that nondegenerate holomorphic maps $f : X \to \mathbb{P}^n$ are in one-to-one correspondence with line bundles $L \to X$ together with basepoint-free linear systems $V \subset H^0(X; L)$ such that $\dim V = n + 1$.

Remark 279. If $V = H^0(X;L)$ we call it a complete linear system. If there exists a basepoint-free linear system $V \subset H^0(X;L)$ we say that L is basepoint-free (or **generated by global sections**).

Example 280. For $X = \mathbb{P}^n$ we have that $L = \mathcal{O}_{\mathbb{P}^n}(k)$ is basepoint-free for all $k \geq 0$. Indeed we can show this explicitly: $H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(k))$ recall is the space of homogeneous polynomials of degree k in $x_0, \ldots, x_{n+1}$. In particular this yields a map

$$\mathbb{P}^n \to \mathbb{P}^N, \qquad N = \binom{n+k}{n} - 1$$

sending a point $(x_0 : \cdots : x_n)$ to all monomials of degree k in $x_0, \ldots, x_n$. This is clearly basepoint free. This map in fact turns out to be an embedding, and is called the **Veronese embedding of** $\mathbb{P}^n$ **of order** k. Notice that one might care about this embedding because it writes hyperplanes of degree k in $\mathbb{P}^n$ as a linear subspace of $\mathbb{P}^N$!

Let us now state the theorem we wish to prove in terms of the language that we have just developed.

Theorem 281 (Kodaira embedding theorem)**.** *Let $L \to X$ be a positive line bundle on a compact complex manifold X. Then there exists a $k_0 \in \mathbb{Z}_{>0}$ such that for each $k \geq k_0$, the line bundle $L^{\otimes k}$ is basepoint-free. Moreover, any induced map $\phi_{L^{\otimes k}} : X \to \mathbb{P}^n$ is an embedding.*

The next step is now to ask when the map defined using linear systems is an embedding. Say $M \to X$ is a line bundle that is basepoint-free, i.e. we have an associated map $\phi_M : X \to \mathbb{P}^n$.

Lemma 282. *The map ϕ_M is an embedding if and only if ϕ_M is injective and for all $x \in X$ the map $d\phi_{M,x} : T'_x X \to T'_{\phi_M}(x) \mathbb{P}^n$ is injective.*

Proof sketch. As X is a map of compact manifolds, ϕ_M is an open mapping whence the first condition implies that ϕ_M is a homeomorphism onto its image. The second condition via the implicit function theory (in the complex case) shows that there exists a holomorphic ϕ_M^{-1} defined on the image. □

We would like to translate these properties to some statements about properties of sections of line bundles. So let $H^0(X;M) = \langle s_0, \ldots, s_N \rangle$. After this choice we have a map

$$\phi_M(x) = (s_0(x) : \cdots : s_N(x)).$$

For ϕ_M to be injective we must have that for each $x \neq y$ on X the vectors $(s_0(x), \ldots, s_N(x))$ and $(s_0(y), \ldots, s_N(y)))$ are linearly independent. We could rephrase this as: the map

$$\mathrm{ev}_x \oplus \mathrm{ev}_y : H^0(X, M) \to M_X \oplus M_y$$

is surjective. This in turn is equivalent to the existence of a section $s \in H^0(X;M)$ such that $s(x) = 0$ and $s(y) \neq 0$. This condition is sometimes phrased that M **separates points**.

Next let's look at the requirement for $d\phi_M$ to be injective. Fix $x \in X$ and assume that $s_0(x) \neq 0$. Locally $s_j = f_j \cdot s_0$ for f_j a holomorphic function for each j. By rescaling, write

$$\phi_M(x) = (1 : f_1(x) : \cdots : f_N(x)).$$

For the differential to be injective at x is to say that the matrix $\partial f_j/\partial x_i$ where $j = 1, \ldots, N$ and $i = 1, \ldots, n$ (recall $n = \dim X$) has maximal rank (rank n). This is of course equivalent to asking that $\langle df_1, \ldots, df_n \rangle$ span the holomorphic cotangent space $T^{1,0}_x X$. More intrinsically, consider the following. Let $\mathcal{I}_x \subset \mathcal{O}_X$ be the ideal sheaf of holomorphic functions vanishing at x. This yields an inclusion

$$H^0(X; M \otimes \mathcal{I}_x) \subset H^0(X; M)$$

where the left is now the global sections of M vanishings at x. We have a section $t \in H^0(X; M \otimes \mathcal{I}_X)$ which we can write as $t = f \cdot t_0$ where $t_0(x) \neq 0$ and $f(x) = 0$. This yields a map

$$H^0(X; M \otimes \mathcal{I}_X) \to T^{1,0}_x X \otimes M_x$$

which is surjective if and only if the differential $d\phi_M$ is injective at x. This is often phrased as **separation of tangent vectors**.

We're out of time but try to go through these technical details on your own as well.

37. April 23, 2018

37.1. **Cohomological criteria.** Recall from last time that if M is a line bundle on X then

- M is basepoint-free at $x \in X$ if and only if $\mathrm{ev}_X : H^0(X; M) \to M_x$ is surjective;
- M additionally yields an embedding $\phi_M : X \to \mathbb{P}^N$ if it separates points as well as tangent vectors;
- M separates points if $\mathrm{ev}_x \oplus \mathrm{ev}_y : H^0(X; M) \to M_X \oplus M_Y$ is surjective;
- M separates tangent vectors if the map $H^0(X; M \otimes \mathcal{I}_X) \to T_x^{1,0} X \otimes M_x$ is surjective.

There is always an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_X \to \mathcal{O}_x \to 0$$

which, upon tensoring with M and passing to cohomology, yields

$$0 \to H^0(X; M \otimes \mathcal{I}_X) \to H^0(X; M) \xrightarrow{\mathrm{ev}_x} M_x \to H^1(X; M \otimes \mathcal{I}_x) \to \cdots$$

In particular we see that if $H^1(X; M \otimes \mathcal{I}_x) = 0$ then ev_x is surjective, i.e. M is basepoint-free. How do we get vanishing results on this cohomology? It would be great if this sheaf $M \otimes \mathcal{I}_x$ were a line bundle. This is true if X is a curve, but more generally let us force x to turn into a divisor – we will blow up at the point x to instead work with a line bundle (so that we might apply KAN vanishing, say).

37.2. **Blowups.** Let $\dim X = n$. Recall that we have $\pi : \tilde{X} = \mathrm{Bl}_x X \to X$. Recall that the fiber of this map is a single point everywhere except for at the point x. Here the fiber is the exceptional divisor $E \cong \mathbb{P}^{n-1}$, the space of local directions at x. Since E is a hypersurface, we obtain a line bundle $\mathcal{O}_{\tilde{X}}(-E)$. We also have, on E itself, the line bundle $\mathcal{O}_E(1) := \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. These two are related essentially by the construction of the blowup.

Lemma 283. *We have that $\mathcal{O}_E(E) := \mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_E(1)$.*

Proof. Since the statement is one about sheaves on E we may assume that $X = \mathbb{C}^n$. Then we have $\pi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$, where recall the blowup is an incidence correspondence $\tilde{\mathbb{C}}^n \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. Denote the projection onto the second factor by q. We saw long ago that $\tilde{\mathbb{C}}^n$ (or q) is the total space of the line bundles $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Any bundle is of course equipped with the zero section, let us call it s. The image of the zero section is precisely the exceptional divisor E.

We now claim that $\mathcal{O}_{\tilde{\mathbb{C}}^n}(-E) \cong q^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. We leave this as a homework exercise: given a line bundle $L \to X$ with zero section s whose image we write E, then $\mathcal{O}_L(E) \cong q^*L$ where on the right we think of L as a sheaf. This can be thought of as an isomorphism of line bundles on L or sheaves of $\mathcal{O}_L$-modules.

Dualizing this result and restricting to E we exactly obtain $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. □

Lemma 284. *The canonical bundle of the blowup is computed $\omega_{\tilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$.*

Proof. Write $U = X \setminus \{x\} \cong \tilde{X} - E$. Certainly we have $\omega_{\tilde{X}}|_U \cong \pi^*\omega_X|_U$. Recall we have a standard open cover of $\tilde{X}$ given by $V_j = q^{-1}U_j$ where U_j is a standard open set in $\mathbb{P}^{n-1}$. In coordinates recall that the blowup is deifned as points (x, z) where $x_i z_j = x_j z_i$. We have a trivialization $V_j \to \mathbb{C}^n$ given

$$(x, z) \mapsto \left(\frac{x_1}{x_j}, \cdots, \frac{\hat{x_j}}{x_j}, \cdots \frac{\hat{x_n}}{x_j}, z_j \right).$$

Next we have $V_j \to \pi(V_j)$

$$(y_1, \dots, y_n) \mapsto (y_j y_1, \dots, y_j y_{j-1}, y_j, y_j y_{j+1}, \dots, y_j y_n)$$

and $E \cap V_j = (y_h = 0)$ in this chart.

Now we compute

$$\pi^*(dz_1 \wedge \cdots \wedge dz_n) = y_j^{n-1} dy_1 \wedge \cdots \wedge dy_n.$$

This shows us that the twist is by the $(n-1)$st tensor power of the line bundle associated to E. □

37.3. **Towards Kodaira embedding.** Today let's focus on showing that $L \to X$ positive on X compact complex implies that some large enough power of L is basepoint-free. In other words we wish to show that $\mathrm{ev}_x : H^0(X; L^{\otimes k}) \to L_x^{\otimes k}$ is surjective. To do this we wish to show a certain first cohomology group vanishes. We first pass to a blowup. Write $\pi : \tilde{X} = \mathrm{Bl}_x X \to X$ and $\tilde{L} = \pi^* L$.

We first claim that for all k

$$H^0(X; L^{\otimes k}) \xrightarrow{\pi^*} H^0(\tilde{X}; \tilde{L}^{\otimes k})$$

is an isomorphism. Injectivity is clear: if $s \circ \pi$ is the zero section then s is the zero section. Now for surjectivity let $\tilde{s} \in H^0(\tilde{X}, \tilde{L}^k)$. We have that $\tilde{s}|_U$ is viewing as a section of $L^{\otimes k}|_U$. If $n = 1$ blowup has done nothing so we are done, but if $n \geq 2$ then the codimension of $\{x\} \subset X$ is at least 2 whence Hartog's lemma allows us to extend this section to all of X.

Remark 285. If you know a bit more about blowups you might use the projection formula, for instance.

Our next claim is that

$$H^0(X; L^{\otimes k} \otimes \mathcal{I}_x) \cong H^0(\tilde{X}; \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)).$$

The left is global sections downstairs vanishing at x and the right is sections vanishing along E. But under the pullback map these are precisely the same.

Next recall our basic sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X; L^{\otimes k} \otimes \mathcal{I}_x) & \longrightarrow & H^0(X; L^{\otimes k}) & \xrightarrow{ev_x} & L_x^{\otimes k} & \longrightarrow & H^1(X; L^{\otimes k} \otimes \mathcal{I}_x) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(X; \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)) & \longrightarrow & H^0(\tilde{X}; L^{\otimes k}) & \longrightarrow & H^0(E; \tilde{L}^{\otimes k}|_E) & \longrightarrow & H^1(\tilde{X}; \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)) \end{array}$$

The first two vertical arrows are isomorphisms by the claims above. But $H^0(E; L^{\otimes k}|E) \cong H^0(E; \mathcal{O}_E) \cong \mathbb{C}$ whence the third is also an isomorphism.

We conclude that for some k, $L^{\otimes k}$ is basepoint-free at x if

$$H^1(\tilde{X}; \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)) = 0.$$

This will now be related to positivity fairly straightforwardly by KAN vanishing.

38. April 25, 2018

38.1. **Proving Kodaira embedding.** Recall our setup. Let X be a compact complex manifold and $L \to X$ be a positive line bundle on X. Last time we fixed $x \in X$ and considered $\pi : \tilde{X} = \mathrm{Bl}_x X \to X$, writing $\tilde{L} = \pi^* L$. Last time we showed that $L^{\otimes k}$ is basepoint-free at x if

$$H^1(\tilde{X}; \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)) = 0.$$

We want to show that there exists $k_0 > 0$ such that L^k is basepoint-free for $k \geq k_0$. We have only one tool: KAN vanishing. Notice first that

$$\tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E) = \omega_{\tilde{X}} \otimes \omega_{\tilde{X}}^{-1} \otimes \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E).$$

Define M_k by $\tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E) = \omega_{\tilde{X}} \otimes M_k$. Using the result on the canonical bundle of the blowup from last time we find that

$$M_k \cong \pi^*(L^{\otimes k} \otimes \omega_X^{-1}) \otimes \mathcal{O}_{\tilde{X}}(-nE).$$

We want M_k to be positive for k sufficiently large; write $k = k_0 + nm_0$,

$$M_k \cong \pi^*(L^{\otimes k_0} \otimes \omega_X^{-1}) \otimes \left(\tilde{L}^{\otimes m_0} \otimes \mathcal{O}_{\tilde{X}}(-E)\right)^{\otimes n}.$$

For k_0 sufficiently large, $L^{otimes k_0} \otimes \omega_X^{-1}$ is positive. Of course this has nothing to do with ω_X^{-1} – we could replace it with any line bundle and there'd always be a large enough k_0. Indeed, $k_0 h_L$ dominates any other metric since our manifold is compact.

The key lemma is the following.

Lemma 286. *If L is positive then $\tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)$ is positive for $k \gg 0$.*

Suppose for the moment that the lemma is true. Then the pullback term in M_k is nonnegative for large enough k_0 and the rest is positive for large enough m_0 by the lemma. Applying KAN vanishing, we are done (with the basepoint-free part of the proof).

Proof. We wish to produce a positive metric on $L^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)$. On $\tilde{L}$ we have the Hermitian metric induced by the metric h_L on L. If we write $c_1(L) = [\omega]$ where $\omega = i/2\pi \cdot \Theta_L$, then $\pi^*\omega = i/2\pi \cdot \pi^*\Theta_L = i/2\pi \cdot \Theta_{\tilde{L}}$. We know that $\pi^*\omega$ is positive on $\tilde{X} \setminus E \cong X \setminus \{x\}$. Moreover $\tilde{L}|_E = \pi^* L|_E \cong \mathcal{O}_E$ and $\pi^*\omega \equiv 0$ along E.

Locally around $x \in X$ we choose coordinates $z_1, \ldots, z_n$ on an open $U \subset X$ where $U \cong D \subset \mathbb{C}^n$ is a ball. We have that $U_1 := \pi^{-1}(U) \cong \mathrm{Bl}_0(D) \subset D \times \mathbb{P}^{n-1}$. We have a projection $q : U_1 \to \mathbb{P}^{n-1}$ and we know from last time that $\mathcal{O}_{U_1}(-E) \cong q^*\mathcal{O}(\mathbb{P}^{n-1})(1)$ on which we get the metric h_1 pulled back from the (anti)tautological bundle $\mathbb{P}^{n-1}$. Moreover

$$c_(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = [\omega_{\mathsf{FS}}], \qquad \frac{i}{2\pi}\Theta_1 = q^*\omega_{\mathsf{FS}}.$$

Write $U_2 = \tilde{X} \setminus E$ and notice that $\mathcal{O}_{\tilde{X}}(-E)$ is trivial on U_2. Choose a trivializing section s_E and a metric h_2 such that $h_2(s_E, s_E) = 1$. We now have that $\tilde{X} = U_1 \cap U_2$ and metrics on each of these opens, so we do the usual thing. Choose a partition of unity subordinate to this decomposition and define

$$h_E = \rho_1 h_1 + \rho_2 h_2.$$

It is an exercise to check that this yields a metric on $\mathcal{O}_{\tilde{X}}(-E)$.

Recall that we want to show that $\tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)$ is positive for k large enough. Consider

$$\eta = \frac{i}{2\pi}\Theta_{\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)} = k\pi^*\omega + \frac{i}{2\pi}\Theta_E$$

Since $U_1 \subset U \times \mathbb{P}^{n-1}$ we have that

$$\eta = \frac{i}{2\pi}\Theta_{\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)} = k\pi^*\omega + q^*\omega_{\mathsf{FS}}$$

which is positive (notice that the individual pullbacks are not positive! only the external product!). We conclude that η is positive on U_1. Next fix a small neighborhood V of E in $\tilde{X}$. On $\tilde{X} \setminus V \subset U_2$ we have that $i/2\pi\theta_E$ (like in the argument above) is bounded by compactness but $k\pi^*\omega$ is positive since we are away from E. □

What was special here was that $\mathcal{O}_E(E)$ is positive since it is coming from $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Let's now return to Kodaira embedding. We haven't quite proven freeness yet. We have shown that for each $x \in X$ there exists k_x such that L^k is basepoint-free at x for all $k \geq k_x$. We can globalize and choose one k by the following argument: if a bundle is basepoint-free at a point it is basepoint-free in an open neighborhood since freeness is a nonvanishing condition. Now simply apply compactness (take a maximum over a finite number of neighborhoods).

So far all we know is that there is a map to projective space

$$\phi_{L^k} : X \to \mathbb{P}^N \quad k \geq k_0.$$

It remains to check that this map separates points as well as tangent vectors.

We first consider the separation of points. We now take $\pi : \tilde{X} = \mathrm{Bl}_{\{x,y\}} X \to X$; we want the surjectivity of $\mathrm{ev}_x \oplus \mathrm{ev}_y : H^0(X; L^k) \to L^k_x \oplus L^k_y$. Precisely as in the case of one point, this surjectivity is equivalent to the surjectivity of

$$H^0(\tilde{X}; \tilde{L}^k) \to H^0(E_x, \tilde{L}^k|_{E_x}) \oplus H^0(E_y; \tilde{L}^k|_{E_y}),$$

which sits in a long exact sequence. In particular we obtain surjectivity if

$$H^1(\tilde{X}; \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E_x - E_y) = 0.$$

Now follow the same argument as before. However, there is one important problem in globalizing to $(x, y) \in X \times X \setminus \Delta_X$, as this space is not compact. Let's come back to this subtlety.

We now consider the separation of tangent vectors. We want to show that for k sufficiently large and for every $x \in X$, the map

$$H^0(X; L^k \otimes \mathcal{I}_{\{x\}}) \to T^{1,0}_x X \otimes L^k_x$$

is surjective. Yet again we wish to apply KAN vanishing so we need to pass to the blowup. We claim that this map can be rewritten as

$$H^0(\tilde{X}; \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \to T^{1,0}_x X \otimes L^k_x \cong H^0(E, \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}})(-E)|_E).$$

The long exact sequence we are interested in comes from tensoring

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0$$

with $\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$. We write out some terms:

$$H^0(\tilde{X}, \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \to H^0(E; \tilde{L}^k \otimes \mathcal{O}(-E)|_E) \to H^1(\tilde{X}; \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-2E))$$

so it is sufficient for this first cohomology to vanish. Now the argument will follow very similarly to above. For globalizing we will use compactness (but notice that injectivity of the differential at a point implies injectivity in a small neighborhood!) We'll finish up next time.

39. April 27, 2018

39.1. **Tying up loose ends.** Last time we showed that if $x \neq y \in X$ then there exists a $k = k(x, y)$ such that L^k separates x and y. Moreover, given $x \in X$ then there exists $k = k(x)$ such that L^k separates tangent vectors, i.e.

$$H^0(X; L^k) \to T_x^{1,0} X \oplus L_x$$

is surjective. We claim that $T_x^{1,0} X \otimes L_x \cong H^0(\tilde{E}; \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)|_E)$. This is because

$$\tilde{L}^k|_E \otimes \mathcal{O}_{\tilde{X}}(-E)|_E \cong \pi^* L_x^k \otimes \mathcal{O}_E(1)$$

whence

$$H^0(X; L^k \otimes \mathcal{I}_x) \to T_x^{1,0} X \oplus L_x \cong L_x \otimes H^0(E; \mathcal{O}_E(1)) \cong L_x^k \otimes T_x^{1,0} X.$$

This last isomorphism follows from thinking about the exceptional divisor as parametrizing the projectized tangent space at x. We conclude that it is enough to have (from the long exact sequence from last time)

$$H^1(\tilde{X}; \tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-2E)) = 0.$$

Now the argument proceeds as before: we know that $\tilde{L}^p \otimes \mathcal{O}_{\tilde{X}}(-E)$ is positive for $p \gg 0$ whence $\tilde{L}^{2p} \otimes \mathcal{O}_{\tilde{X}}(-2E)$ is positive for $p \gg 0$. Apply Kodaira vanishing as before, and we are done (extending to a uniform bound for k over X by the usual compactness argument).

In fact, however, if we separate tangent vectors we know that $d\phi_{L^k}$ is injective for each x, which implies that ϕ_{L^k} is injective locally around x. We know that for each point not on $\Delta_X \subset X \times X$ there is a neighborhood where points are separated. But the same argument holds for points on the diagonal: we can find a neighborhood of points on the diagonal for which we have separation of points. Thus we can extract a finite subcover and take maximums to find a k such that L^k separates $(x, y) \in X \times X$. This completes the proof of Kodaira's embedding theorem.

39.2. **Corollaries of Kodaira embedding.** Algebraic geometers often prefer to use the notion of ampleness.

Definition 287. We say that a line bundle L is **ample** if ϕ_{L^k} is an embedding for $k \gg 0$. We say that L is **very ample** if ϕ_L is an embedding.

Corollary 288 (Restatement of Kodaira embedding)**.** *L is positive if and only if L is ample.*

Corollary 289. *If X is projective then $\mathrm{Bl}_x X$ is projective.*

Proof. We saw during the course of the proof of Kodaira embedding that if L is positive then $\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$ is positive for $k \gg 0$. Hence $\mathrm{Bl}_x X$ has a positive line bundle so the result follows from Kodaira embedding. □

Corollary 290 (Another restatement of Kodaira embedding)**.** *If X is a compact Kähler manifold then X is projective if and only if there exists a two-form $\omega \in A^2 X$ that is a closed positive $(1,1)$-form such that $[\omega] \in H^2(X; \mathbb{Q}) \subset H^2(X; \mathbb{R})$.*

Proof. If X is projective then take $\omega = \omega_{\mathsf{FS}}$ whence $[\omega] \in H^2(X;\mathbb{Z})$. Conversely, given $[\omega] \in H^2(X;\mathbb{Q})$ there exists some k such that $[k\omega] \in \operatorname{im} H^2(X;\mathbb{Z})$. This is of course still a $(1,1)$-form. Now the Lefschetz $(1,1)$-theorem tells us that there exists a line bundle L such that $c_1(L) = [k\omega]$. But now L is positive by construction so we apply Kodaira embedding theorem to find that X is projective. □

The following result shows, for instance, that all Calabi-Yaus not of dimension two are projective.

Corollary 291. *Let X be compact Kähler with $H^2(X;\mathcal{O}_X) = 0$. Then X is projective.*

Proof. Let h_0 be the Kähler metric with $(1,1)$-form ω_0 closed and positive. We know that $[\omega_0] \in H^2(X;\mathbb{R})$. We see that

$$0 = H^2(X;\mathcal{O}_X) = H^{0,2}X = H^{2,0}X,$$

whence $H^2(X;\mathbb{C}) = H^{1,1}X$ by the Hodge decomposition. But notice that we have an inner product on $H^2(X;\mathbb{R})$ coming from h_0, denote it $\langle \alpha, \beta \rangle_X$. But $H^2(X;\mathbb{Q})$ is the space of rational points of $H^2(X;\mathbb{R})$ and hence is dense (these spaces are finite-dimensional). In other words, for each $\varepsilon > 0$ there exists a rational $(1,1)$-form ω, say harmonic, with $\|\omega - \omega_0\| < \varepsilon$. But ω_0 is positive and X is compact (so the difference is bounded) whence ω is positive. Hence X is projective. □

The following is immediate for dimensional reasons.

Corollary 292. *Every compact Riemann surface is projective.*

Definition 293. We say that X is **Calabi-Yau** if $\omega_X \cong \mathcal{O}_X$ and $H^i(X;\mathcal{O}_X)$ for all $0 < i < \dim X$.

You will see varying definitions depending on how one wants to use the Calabi-Yau but this is pretty standard. Sometimes one asks that X be simply-connected. Under this definition we obtain the following.

Corollary 294. *Every Calabi-Yau n-fold with $n \geq 3$ is projective.*

The two-dimensional case is rather special.

Definition 295. A Calabi-Yau of dimension 2 is called a **K3 surface**.

It turns out that there exist nonprojective K3 surfaces! More specifically, there exists a 20-dimensional moduli space of all K3 surfaces. Inside this space there is a countable union of 19-dimensional closed analytic subspaces that parametrize the projective K3 surfaces.

40. April 30, 2018

(I think we assume definitionally that K3 surfaces are simply-connected.)

Example 296. Consider $X \subset \mathbb{P}^3$ a quartic surface, i.e. we have $F \in \mathbb{C}[X_0, \ldots, X_3]$ homogeneous of degree 4. The space of such is parameterized by $H^0(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(4))$ which has dimension $\binom{4+3}{3} = 35$. Generically such a polynomial yields a manifold. Notice that the zero locus of f is also the zero locus of λf. Hence the quartic K3 surfaces (seen as embedded in $\mathbb{P}^3$) are an open set in $\mathbb{P}(H^0(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(4)))$. Now we have to take into account the automorphisms of projective space (recall that this is $PGL_4(\mathbb{C})$: given $\phi : \mathbb{P}^3 \to \mathbb{P}^3$, the locus $\phi(X_1)$ is again a quartic K3. This is all we have to take into account. We conclude that the isomorphism classes of quartic K3 surfaces is in bijection with $U/PGL_4(\mathbb{C})$ which has dimension $34 - 15 = 19$.

I didn't quite catch/understand this argument

Remark 297. In fact, most K3 surfaces are not algebraic. The set of all K3 surfaces forms a 20-dimensional family while the projective K3s form a (countable family of) 19-dimensional family.

Let X be a compact complex manifold. There exists a space of possible complex structures. The "infinitesimal deformations of complex structure" are parameterized by $H^1(X; T_X)$. One has to be a bit careful because not all points in the space of complex structures are smooth. We will hopefully prove this rigorously later.o

Let us now specialize to the case that X is a K3 surface. We have that $\omega_X \cong \mathcal{O}_X$. This implies that $H^1(X; T_X) \cong H^1(X; \Omega^1_X)$. This is because there is a perfect pairing of Ω^1_X with itself to the trivial bundle. But $\dim H^1(X; \Omega^1_X) = h^{1,1}$ which we know for every K3 surface is 20. We haven't justified this yet but we will. Hence we conclude that there should be a 20-dimensional family of K3s.

Why are the projectives 19-dimensional? Well suppose we are at a point in the moduli space that is X_0 a projective K3. There exists a ω_0 closed $(1,1)$-form $[\omega_0] \in H^2(X_0; \mathbb{Z}) \cap H^{1,1}X$. For some X close to X_0 the singular cohomology groups are of course isomorphic, and under this isomorphism $[\omega_0] \mapsto [\omega]$. If we now use the Hodge decomposition,

$$H^2(X_0; \mathbb{C}) \cong H^2(X; \mathbb{C}) = H^{2,0}X \oplus H^{1,1}X \oplus H^{0,2}X$$

we see that for X to be projective we need that $[\omega_0] \in \ker(H^{1,1}X_0 \to H^{2,0}X)$, to guarantee that we obtain a $(1,1)$-form. But from the Hodge diamond we know that the target is one-dimensional and the source is twenty-dimensional. Hence the projectives form 19-dimensional families.

40.1. Complex tori and Kodaira embedding.

Let's now look at complex tori. The following is a reinterpretation of Kodaira embedding.

Proposition 298 (Riemann's criterion)**.** *If $T = \mathbb{C}^n/\Lambda$ is a complex torus then T is projective if and only if there exists a positive-definite Hermitian form $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that the imaginary part of h takes integral values on $\Lambda \times \Lambda$. In this case we call T an* ***abelian variety****.*

Proof. Write $V_{\mathbb{R}} = H^1(T; \mathbb{R})$ and decompose the complexification $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. On $V^{1,0}$ we can choose a basis $dz_1, \ldots, dz_n$ spanning a vector space isomorphic to $T_0^*\mathbb{C}^n \cong (\mathbb{C}^n)^*$ using the natural identification $T_0\mathbb{C}^n \cong \mathbb{C}^n$. Recall now that $V^{p,q} \subset H^{p,q}(T; \mathbb{C})$ is $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ In particular, $(1,1)$-forms are in correspondence with $\omega \in V^{1,1} = V^{1,0} \otimes V^{0,1}$ We have identified $V^{1,0} \cong (\mathbb{C}^n)^*$ and $(\overline{\mathbb{C}^n})^*$. This is of course the choice of an Hermitian form $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$.

Now from all the algebra we did at some point, ω is identified as $-\operatorname{im} h$ and ω is positive-definite if and only if h is positive-definite.

Finally we think about the integrality. First recall that $H_1(T;\mathbb{Z}) \cong \Lambda$. By the universal coefficient theorem, $V_{\mathbb{R}} \cong \operatorname{Hom}_{\mathbb{Z}}(\Lambda\mathbb{R})$ whence

$$\Lambda^i V_{\mathbb{R}} \cong \operatorname{Hom}_{\mathbb{Z}}(\Lambda^i \Lambda, \mathbb{R})$$

Consider the integral points of this vector space: $\operatorname{Hom}_{\mathbb{Z}}(\Lambda^i\Lambda, \mathbb{Z}$. Hence ω is integral if and only if the corresponding element is in $\operatorname{Hom}_{\mathbb{Z}}(\Lambda^2\Lambda, \mathbb{Z})$. This element corresponds to the Hermitian form. ☐

Notice that most tori are not projective! In fact, its worse than that. Most tori do not have any nontrivial analytic subvarieties in general. What does this have to do with projectivity? If one is embedding in $\mathbb{P}^n$ one can intersect with hypersurfaces to obtain subvarieties that are neither X nor points. That's why it seems rather peculiar to not have nontrivial subvarieties.

Example 299. There exists a 2-dimensional torus such that the only subvarieties are T and points. Consider $V = \mathbb{C} \oplus \mathbb{C}$ and let $J : V \to V$ be the map sending $(z, w) \mapsto (iz, -iw)$. We want a lattice $\Lambda \subset V$ such that $J(\Lambda) = \Lambda$. For instance one can define Λ by the choice of basis vectors v_1, v_2, Jv_1, Jv_2, i.e. by the matrix

$$\begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix}$$

such that $a, b, c, d \in \mathbb{C}$ and the 4 vectors are linearly independent. In particular we obtain a 4-parameter family of such Λ. Write e_i for the columns of this matrix.

Lemma 300. *Let $f = a\bar{d} - b\bar{c}$. If $\theta = f^{-1}dz \wedge d\bar{w}$ then the real and imaginary parts of θ are closed $(1,1)$-forms with class in $H^2(T;\mathbb{Z})$.*

Proof. Left as an exercise. For the integrality, if we call the real and imaginary parts α and β then we find that

$$\alpha = e_1^* \wedge e_2^* - e_3^* \wedge e_4^*$$
$$\beta = e_1^* \wedge e_4^* - e_2^* \wedge e_3^*.$$

☐

Lemma 301. *If Λ is generic in this four-parameter family then $H^2(T;\mathbb{Z}) \cap H^{1,1}T = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$.*

Proof. This is again just a (long) calculation. By generic we mean that all the choices of a, b, c, d that mess up this condition are proper analytic subvarieties of all choices. ☐

We now compute

$$J\theta = f^{-1} idz \wedge id\bar{w} = -\theta.$$

Moreover $J\alpha = -\alpha$ and $J\beta = -\beta$. Hence for generic Λ $J\gamma = -\gamma$ for each $\gamma \in H^2(T;\mathbb{Z}) \cap H^{1,1}T$. Now let's say we have a curve $C \subset T$ on this torus (we may assume that C is a manifold). Now we obtain the Poincaré dual $\eta_C \in H^2(T;\mathbb{Z}) \cap H^{1,1}T$. We have that $J\eta_C = -\eta_C$ but $J\eta_C = \eta_{J(C)}$ since J preserves Λ whence yields an automorphism of the torus. But this is a contradiction: if ω is the standard Kähler form

$$0 = \int_T \omega \wedge (\eta + \eta_{J(c)}) = \int_C \omega + \int_{J(c)} \omega = \operatorname{vol}(C) + \operatorname{vol}(J(C)) > 0$$

where we use in the second equality Poincaré duality. Hence there can be no curve in T.

Notice this shows that the Hodge conjecture cannot hold for nonprojective spaces! In volume 2 of Shavarevich's Basic Algebraic Geometry, there are many such examples in chapter 8 section 1.4.

41. May 2, 2018

41.1. **Chow's theorem.** We say that $X \subset \mathbb{P}^n$ is algebraic if there exist homogeneous polynomials $F_1, \dots, F_K \in \mathbb{C}[x_0, \dots, x_n]$ such that $X = Z(F_1, \dots, F_k)$. Recall that F is homogeneous of degree d if $F(\lambda x) = \lambda^d F(x)$.

Theorem 302 (Chow's theorem)**.** *If $X \subset \mathbb{P}^n$ is an analytic subset then X is algebraic.*

Together with Kodaira's embedding theorem we conclude that if X is a compact complex manifold with a positive line bundle, then X is algebraic.

Before we prove Chow's theorem, let us make some additional comments. This theorem has been expanding over the years to very general statements (still quite classical, though!). For instance there is the vast generalization of GAGA due to Serre. This states that (over $\mathbb{C}$) the category of coherent analytic sheaves on projective varieties is equivalent to the category of coherent algebraic sheaves. We haven't really discussed coherent sheaves in this class, but you may as well just think of vector bundles.

Example 303. Consider for instance line bundles on $\mathbb{P}^n$. The exponential sequence gave us

$$0 = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to H^1(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}^\times) \to H^2(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z} \to H^2(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}) = 0$$

from which we conclude that the only line bundles are $\mathcal{O}_{\mathbb{P}^n}(k)$ for $k \in \mathbb{Z}$. Everything about these line bundles is algebraic: the trivializations and the transition functions are made of polynomials. However notice that the exponential sequence makes no sense in the Zariski topology! Indeed, recall that $H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(k))$ is just the homogeneous polynomials of degree k in $(n+1)$ variables, even though we are taking holomorphic sections.

So how do you prove something is a polynomial in complex analysis? Here we will use a trick that relies on the following theorem from several complex variables.

Theorem 304 (Levi extension theorem)**.** *Let X be a connected complex manifold of dimension n and let $Z \subset X$ be an analytic subset of codimension at least $k+1$. Let $Y \subset X \setminus Z$ be an analytic subset of codimension k. Then the closure $\bar{Y} \subset X$ is analytic.*

Example 305. For instance consider Hartog's theorem, which says that if $p \in X$ and $\dim X \geq 2$ then if $f \in \mathcal{O}(X \setminus p)$ then f extends to $\tilde{F} \in \mathcal{O}(X)$. Levi's theorem in this context says that if $Y \subset X \setminus p$ is analytic of dimension at least 1, then $\bar{Y}$ is still analytic.

The proof of Chow's theorem is a very simple but smart application of Levi extension.

Proof of Chow's theorem. Let $Z \subset \mathbb{P}^n$ be an analytic subset. Recall that $\mathbb{P}^n$ is the set of linear subspaces of $\mathbb{C}^{n+1}$. We denote by Y the preimage of Z under the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$:

$$Y = \pi^{-1}Z.$$

Notice that every component of Y has dimension at least 1. Y is clearly analytic whence by Levi extension $\bar{Y}$ is analytic. Notice that $\bar{Y}$ is simply adding the origin – we obtain the cone over Z.

Write $\mathcal{O}_{n+1}$ for the ring of germs of holomorphic functions at 0. Consider the ideal $\mathcal{I} \subset \mathcal{O}_{n+1}$ of germs vanishing on $\bar{Y}$. Take $f \in \mathcal{I}$. As f is an analytic function we can write it as

$$f(z) = \sum_{j \geq 0} f_j(z)$$

where the f_j are homogeneous polynomials of degree j. If we fix $x \in \bar{Y}$ notice that $\lambda x \in \bar{Y}$ as well if we take $\lambda > 0$ small enough ($\bar{Y}$ is a cone!). Hence

$$0 = f(\lambda x) = \sum_{j \geq 0} \lambda^j f(x).$$

But now $f(\lambda x)$ is holomorphic in λ whence this is an expansion near $\lambda = 0$. We conclude that $f_j(x) = 0$ for all j. This yields a huge number of homogeneous polynomials vanishing at x whence $\mathcal{I}$ is generated by homogeneous polynomials. We conclude that, since $\mathcal{O}_{n+1}$ is Noetherian, that $\mathcal{I} = (F_1, \ldots, F_k)$ for some homogeneous polynomials F_i. Finally, since $\bar{Y}$ is a cone we conclude that $\bar{Y} = Z(F_1, \ldots, F_k)$. □

41.2. **Examples of varieties.** We have talked about a lot of generalities, so perhaps for a few lectures let us focus on concrete examples.

Let $X = C$ be a smooth projective curve of genus g. Here by genus we mean the purely topological definition, considering the curve as a Riemann surface. Hopefully you remember that the topological Euler characteristic is given

$$\chi_{\mathsf{top}}(C) = 2 - 2g.$$

This fits into our picture of the Hodge diamond: we must have $h^{1,0}X + h^{0,1}X = 2g$ whence $h^{1,0}X = g$. But

$$H^{1,0}C = H^0(C; \Omega^1_C) = H^0(C; \omega_C).$$

Hence the genus is also the number of linearly independent one-forms on C. It turns out that the genus is convenient for classification of curves.

Example 306 (Genus 0)**.** It is not hard to see that if $g = 0$ then $C \cong \mathbb{P}^1$. In this case we know that $\omega_C = \mathcal{O}_{\mathbb{P}^1}(-2)$ whence the canonical bundle is negative.

Example 307 (Genus 1)**.** If $g = 1$ then we call C an elliptic curve, or a torus. In this case $\omega_C \cong \mathcal{O}_C$ so the canonical bundle is trivial. Moreover there is a one-parameter family of such curves (parameterized, say, by the J-invariant).

Example 308 (Genus greater than 1)**.** In this case it turns out that ω_C is positive: in fact,

$$c_1(\omega_C) = 2g - 2 > 0.$$

In the theory of Riemann surfaces, these would be called hyperbolic. Recall that last time we mentioned that in the parameter space of all Riemann surfaces of genus g the tangent space at a point can be thought of as $H^1(C; T_C)$, which turns out to be $3g-3$ dimensional. We should note that this moduli space is in fact an orbifold, not a manifold.

Let's look at the exponential sequence for such curves:

$$0 \to H^1(C;\mathbb{Z}) \to H^1(C;\mathcal{O}_C) \to H^1(C;\mathcal{O}_C^\times) \to H^2(C;\mathbb{Z}) \to 0$$

Of course since we are in two-dimesions, there is an isomorphism $c_1 = \deg : \mathsf{Pic}(C) to \mathbb{Z}$. We denote the kernel of this map by $\mathsf{Pic}^0(C)$. This kernel is also a cokernel

$$\mathsf{Pic}^0(C) = H^1(C;\mathcal{O}_C)/H^1(C;\mathbb{Z}).$$

But recall our discussion about Hodge structures of weight one! This $\mathsf{Pic}^0(C)$ is a complex torus (and in fact projective!) called the **Picard variety** of C. Notice that $H^1(C;\mathcal{O}_C) = H^{1,0}C$ whence $\dim \mathsf{Pic}^0(C) = g$. In other words the Picard group of curves of genus greater than 1 is an extension of this abelian variety by the discrete group $\mathbb{Z}$.

Classically another object of importance is the dual of this torus, the **Jacobian** of C

$$\mathcal{J} = \overline{H^1(C;\mathcal{O}_C)}^\vee / H^1(C;\mathbb{Z})^\vee = H^0(C;\omega_C)^\vee / H_1(C;\mathbb{Z}).$$

It turns out that the Jacobian is naturally isomorphic to the Picard variety (this is subtle and not always true – we will skip this for now). Let us think about how the first homology sits inside the global sections of the holomorphic one-forms. Fix a point $x_0 \in C$. There is a natural map called the **Abel-Jacobi map**

$$\begin{aligned} C &\to \mathcal{J}(C) \\ x &\mapsto \int_{x_0}^x - \end{aligned}$$

which is well-defined after quotienting by the first homology. In fact it turns out that this map is an embedding.

What if we want to think of the Jacobian conretely as $\mathcal{J}(C) = \mathbb{C}^g/\mathbb{Z}^{2g}$? In other words, choose a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms and say the standard generators $\delta_1, \ldots, \delta_{2g}$ for $H_1(C;\mathbb{Z})$. One constructs a big matrix called the **period matrix** of C

$$\begin{pmatrix} \int_{\delta_1} \omega_1 & \cdots & \int_{\delta_{2g}} \omega_1 \\ \\ \int_{\delta_1} \omega_g & \cdots & \int_{\delta_{2g}} \omega_g \end{pmatrix}$$

to encapsulate this data.

42. May 4, 2018

Today will be mostly examples with not many proofs. The proofs are not difficult.

42.1. **Examples of surfaces.** We begin with the notion of a projective bundle. Suppose we have a vector bundle E of rank r on a complex manifold X. Write $\pi : E \to X$ for the projection. We can projectivize each fiber to obtain $p : \mathbb{P}\, E \to X$. This is a fiber bundle whose total space is $\dim X + r - 1$. Over each fiber we have $\mathcal{O}(1)$ and it turns out that these globalize to yield a line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that $\mathcal{O}_{\mathbb{P}(E)}(1)|_{p^{-1}(x)} = \mathcal{O}(1)$. So this gives us a nice class of line bundles on the projectivized bundle. One might hope that this is more or less all we have; indeed, one can show

$$\mathsf{Pic}(\mathbb{P}\, E) = \mathsf{Pic} X \oplus \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}\, E}(1).$$

We remark that there is an isomorphism of projective bundles $\mathbb{P}\,E \to \mathbb{P}\,F$ over X if and only if there exists a line bundle L such that $E \cong F \otimes L$. The simplest example is of course $\mathbb{P}\,\mathcal{O}_X^{\oplus r} \cong X \times \mathbb{P}^{r-1}$.

Definition 309. A **ruled surface** is a projective bundle $p : \mathbb{P}\,E \to C$ where C is a smooth projective curve and E is a rank 2 vector bundle on C.

When $C = \mathbb{P}^1$ we say that p is a **rational ruled surface**; these surfaces are bimeomorphic to $\mathbb{P}^2$ but not isomorphic to $\mathbb{P}^2$ (indeed they have different Picard groups). In fact, if $C = \mathbb{P}^1$ then all rational ruled surfaces are of the form

$$F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$$

for $n \geq 0$. These are known as Hirzebruch surfaces. In fact F_1 is the blowup of $\mathbb{P}^1$ at a point. This completes the class of surfaces with Kodaira dimension $\kappa(X) = -\infty$: the surfaces $\mathbb{P}^2$ and rational ruled surfaces. Roughly you should think that this means ω_X^{-1} is positive in a certain sense.

Next we turn to surfaces where ω_X is neither positive nor negative, in a certain sense. We have seen abelian surfaces (projective tori of dimension 2) and we have seen a bunch of K3 surfaces. In both of these examples the vague statement above is made precise since $\omega_X = \mathcal{O}_X$. But there are other examples. We can consider bi-elliptic surfaces: surfaces like $X = E \times F/G$, where E and F are elliptic curves and G is a subgroup of the group of translations of E (it is a torus of genus 1 so this makes sense) such that G acts on F with $F/G \cong \mathbb{P}^1$. For instance, we could have $G = \mathbb{Z}/2\mathbb{Z}$ acting by translation on E (acting as translation by a 2-torsion point on E and then by involution $\iota : x \mapsto -x$ on F). One checks that $F/\iota \cong \mathbb{P}^1$.

There is a still another class, known as Enriques surfaces, characterized by $\omega_X^{\otimes 2} \cong \mathcal{O}_X$. This is in fact equivalent to

$$H^0(X; \omega_X) = H^0(X; \Omega_X^1) = 0.$$

We briefly mentioned some time ago the following lemma.

Lemma 310. *If Y is a K3 surface and ι is a fixed-point free involution of Y then Y/ι is Enriques.*

Let's construct an example of where this lemma applies.

Example 311. Let $Q_1, Q_2, Q_3 \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials of degree two (i.e. three quadrics in $\mathbb{P}^2$). Next take $Q_1', Q_2', Q_3' \in \mathbb{C}[x_3, x_4, x_5]$ three other quadrics in another copy of $\mathbb{P}^2$. Define now three new quadrics

$$P_i = Q_i(x_0, x_1, x_2) + Q_i'(x_3, x_4, x_5) \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$$

in $\mathbb{P}^5$. Define $Y = Z(P_1) \cap Z(P_2) \cap Z(P_3) \subset \mathbb{P}^5$. It is not hard to see that if the Q_i, Q_i' are general then Y is a smooth manifold. Indeed, Y is a complete intersection of type $(2,2,2)$ in $\mathbb{P}^5$. We conclude, by our general formula for complete intersections,

$$\omega_Y = \mathcal{O}_Y(2 + 2 + 2 - 5 - 1) \cong \mathcal{O}_Y$$

and by Lefshetz that

$$H^1(Y; \mathcal{O}_Y) = 0.$$

Hence Y is a K3 surface.

Now we need an involution. Consider the map

$$\sigma : \mathbb{P}^5 \to \mathbb{P}^5$$
$$(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5).$$

This map descends to Y since the signs don't matter for quadrics. The fixed locus is clearly $(Z_1 = \{x_0 = x_1 = x_3 = 0\}) \cup (Z_2 = \{x_3 = x_4 = x_5 = 0\})$. These are two copies of $\mathbb{P}^2$ in which the original quadrics live. But if the Q_i are generic then they have no common zeroes in Z_1 and similarly for Q_i' and Z_2. One can of course rephrase this in terms of linear systems and basepointfree-ness. Hence $\iota : Y \to Y$ is fixed-point free. We conclude that $X = Y/\iota$ is an Enriques surfaces. In fact, one can show that a generic Enriques surfaces arises through this funny construction.

Exercise 312. The only possibilities for a complete intersection to be a K3 surface are $(4), (2,3), (2,2,2)$.

This completes the classification of surfaces (up to birational or bimeromorphic equivalence) with "$\omega_X = 0$", i.e. surfaces of Kodaira dimension $\kappa(X) = 0$.

Now there are in fact cases where ω_X is neither positive, negative, nor zero, in the relevant sense. Here $\kappa(X) = 1$. For instance take $X = E \times C$ where E is an elliptic curve and C is a curve of $g \geq 2$. All of these (that are not in the previous two categories above) are **elliptic surfaces**, i.e. there exists a map $p : X \to C$ such that the general fiber is an elliptic curve.

We need a brief aside proving the genus-degree formula.

Remark 313. Fix a smooth projective curve of degree d. Such a curve exists by choosing an appropriate section $H^0\mathcal{O}_{\mathbb{P}^2}(d)$. The adjunction formula from earlier will give us the genus of this curve.

$$\omega_C = \omega_{\mathbb{P}^2}|_C \otimes \mathcal{O}_{\mathbb{P}^2}(C)|_C.$$

We have $\deg \omega_C = 2g - 2$ and the degree of the right is $-3d + d^2$, from which we find that

$$g = \frac{(d-1)(d-2)}{2}.$$

In particular if $d = 3$ then $g = 1$.

There are many such elliptic surfaces. Here is one way (among many) to construct them. Choose a Riemann surface C and take the product $C \times \mathbb{P}^2$. Write the two projections as p_1 and p_2. Define, for any basepoint-free line bundle $M \to C$,

$$L := p_1^* M \otimes p_2^* \mathcal{O}_{\mathbb{P}^2}(3).$$

Choose a general section of L. We claim that its zero locus gives us an (non-product!) elliptic surface. Indeed,

$$L|_{p_1^{-1}(x)} = \mathcal{O}_{\mathbb{P}^2}(3)$$

whence the section along this fiber is a planar cubic, hence an elliptic curve.

There is one giant class of projective surfaces left: those where "ω_X is positive". These are called surfaces of general type. There are of course many examples but they can be quite difficult to write down. Take for instance a trivial example like $C_1 \times C_2$ where $g(C_i) \geq 2$. Or, take like Kollar did in his lectures, any hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. These surfaces are quite hard to classify but they generally have certain nice properties.

43. May 7, 2018

43.1. **Families of manifolds.** We will switch gears now to discuss families of manifolds. We have already seen various spaces that depend on parameters, such as tori, for instance. We will now try to understand such behavior in some generality.

Definition 314. Let $\mathcal{X}$ and B be complex manifolds with B connected. A proper holomorphic map $\pi : \mathcal{X} \to B$ is a **family of compact complex manifolds** with base B and total space $\mathcal{X}$ if π is a submersion.

We often call this a **smooth proper morphism**. So, for each $t \in B$, we write $X_t = \pi^{-1}(t)$ is a compact complex manifold. If we fix $0 \in B$ we call X_t a **deformation** of X_0.

Example 315. Consider $\pi : \mathbb{P}(E) \to X$ is a projective bundle. Then for each t we have noncanonical isomorphisms $X_t \cong \mathbb{P}^{r-1}$ where $r = \operatorname{rk} E$. In fact π is locally trivial, but usually not trivial. Indeed, in general $\mathbb{P}\, E \cong \mathbb{P}\, F$ if there is a line bundle L such that $E \cong F \otimes L$, so trivial if and only if $E = \oplus L$.

Example 316. In $\mathbb{P}^n$ consider the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}\, H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. This is of course isomorphic to $\mathbb{P}^N$ for $N = \binom{n+d}{n}$ and there exists an open set $U \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ of *smooth* degree d hypersurfaces. There exists a universal hypersurface $\mathcal{X} \subset U \times \mathbb{P}^n$. By universal we mean that if we choose a point U corresponding to a hypersurface, the fiber over that point is isomorphic to that hypersurface. Fix a basis $F_0, \dots, F_N$ of smooth hypersurfaces of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and fix coordinates $Y_0, \dots, Y_N$ on $\mathbb{P}^N$. Notice that $U \times \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^n$ so we define

$$\mathcal{X} = Z(Y_0 F_0 + \cdots + Y_N F_N).$$

The family $\pi : \mathcal{X} \to B$ is not even locally trivial! I didn't understand the reasoning. It is trivial to see that U is connected: one has to write down the equations for nonsingulaity and note that it is the complement of a hypersurface. Since smooth hypersurfaces are connected (one can compute cohomologies in exact sequences) the total space $\mathcal{X}$ is connected as well.

Example 317. Elliptic curves are defined by a lattice $\langle 1, \tau \rangle \subset \mathbb{C}$ for τ in the upper half plane. There exists a family $\pi : \mathcal{X} \to \mathbb{H}$.

Let's look at an example of how a map can fail to be a submersion.

Example 318. There exists $\pi : \mathcal{X} \to B$ a map of complex manifolds such that a special fiber is not a manifold (in fact not even an analytic space). Instead, it is a scheme. The issue is that π fails to be a submersion. Consider, for instance, the fibers in $\mathbb{C}^3$ (this example can be compactified to $\mathbb{P}^3$ so one can actually obtain a proper map). Look at $X_a \subset \mathbb{C}^3$ given parametrically:

$$x = t^2 - 1 \qquad y = t^3 - t \qquad z = at$$

for $a \in \mathbb{C}$. The base of our family is $\mathbb{C}$ parameterized by a. For $a \neq 0$ the solution set in $\mathbb{C}^3$ is a twisted cubic (for a picture see Hartshorne's section on flatness) because one can make an appropriate change of variables, something like

$$t = z/a \qquad t^2 = x + 1, \qquad t^3 = y + z/a.$$

The intuition is that the branches of the cubic are coming closer and closer to each other as $a \to 0$ until they intersect.

We are looking for $\mathcal{X} \subset \mathbb{C} \times \mathbb{C}^3$ over $B = \mathbb{C}$. It is easy to check that

$$\mathcal{X} = Z\left(a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)\right)$$

satisfies that $\mathcal{X}|_a = X_a$ is a twisted cubic for $a \neq 0$. The special point is given

$$X_0 = Z\left(z^2, yz, xz, y^2 - x^2(x+1)\right),$$

which is not a manifold. In fact it also contains $z^2 = 0$, so z is a nilpotent element! Hence X_0 is not even a variety since the ring of functions has a nilpotent. Intuitively this nilpotent direction is remembering which "direction" it used to be nonsingular.

We will not be considering such examples here. If we wanted to allow singularities like these we would replace a smooth morphism with a flat morphism.

The picture of smooth proper morphisms for smooth manifolds is much nicer.

Theorem 319 (Ehresmann)**.** *If $\pi : \mathcal{X} \to B$ is a proper submersion of differentiable manifolds then all the fibers are diffeomorphic. In fact, π is locally trivial; more precisely if B is contractible with basepoint $0 \in B$ then there exists a diffeomorphism $\mathcal{X} \simeq X_0 \times B$ commuting with the projection to the base.*

Sketch. Locally on the base, we can connect points with arcs whence we may assume (for the proof of the first statement) we may assume that B is an open interval of $\mathbb{R}$ containing 0 and 1. We want to show that there is a diffeomorphism $X_0 \cong X_1$. Since π is a submersion, for each $x \in \pi^{-1}(0)$, $d\pi_x : T_x X_0 \to T_0 B$ is surjective. We can lift the vector field d/dt to a vector field in a neighborhood U_i of x in $\mathcal{X}$. We can in fact lift to a cover $\{U_i\}$ of X. For t_0 close to 0 we have that $X_{t_0} \subset \cup_{i \in I} U_i$. We now obtain a diffeomorphism $X_0 \cong X_{t_0}$ via the flow of the lift of the vector field. By compactness of $[0,1]$ we iterate over a cover of the interval to obtain a diffeomorphism $X_0 \cong X_1$.

If B is contractible one can show that there exists a vector field on B that lifts to a vector field everywhere whose flow now globally trivializes the family. □

What *can* we say in the holomorphic setting? We have some extra information. Locally we have that diffeomorphisms $\mathcal{X}_U \cong U \times X_0$ over U.

Theorem 320 (Fisher-Grauert)**.** *If the fibers of π are complex manifolds that are isomorphic (as complex manifolds) to each other then π is locally trivial.*

Theorem 321 (Kodaira)**.** *Given a family of complex manifolds, if X_0 is Kähler then there exists an open neighborhood of 0 in B such that each fiber in this neighborhood is Kähler.*

There is a nice classical book of Kodaira called something like deformations of complex structure.

44. May 9, 2018

44.1. **Kodaira-Spencer map.** We will go through 2 different approaches, the first geometric and the second analytic. Recall that we are looking at families $\pi : \mathcal{X} \to B$ of compact complex manifolds. Fix a basepoint $0 \in B$ and denote by X the fiber $X = X_0 = \pi^{-1}(0)$. For every $x \in X$ we can consider the differential $d\pi_x : T_x\mathcal{X} \to T_0B$ where we mean the holomorphic tangent spaces (but do not add notation for typographical reasons). The kernel of $d\pi_x$ is the tangent space to the fiber T_xX. We have a short exact sequence

$$0 \to T_X \to T_{\mathcal{X}}|_X \to N_X|_{\mathcal{X}} \cong T_0B \otimes_{\mathbb{C}} \mathcal{O}_X \to 0$$

since π is a submersion. Globally we could also write $d\pi : T_{\mathcal{X}} \to \pi^*T_B$ and then restrict to X. We now pass to cohomology and obtain a map

$$\kappa : T_0B \otimes_{\mathbb{C}} H^0(X; \mathcal{O}_X) \cong T_0B \to H^1(X; T_X)$$

since X is compact. This map κ is known as the **Kodaira-Spencer map**.

What is this thing and why is it important? The idea is that $v \in T_0B$ leads to an infinitesimal deformation of X (along a tiny arc in the direction of v, say) and $\kappa(v) \in H^1(X; T_X)$ will parameterize the complex structure associated to that deformation. Unfortunately we don't yet have the language to make nilpotent functions on spaces precise.

Definition 322. A **complex space** is a Hausdorff space X together with a sheaf of rings $\mathcal{O}_X$ such that there exists an open cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (Z, \mathcal{O}/\mathcal{J})$ where $Z \subset U \subset \mathbb{C}^n$ is an analytic subset of U open in $\mathbb{C}^n$ and $\mathcal{J} \subset \mathcal{O}_U$ is an ideal sheaf such that $Z = Z(\mathcal{O}_U/J)$.

We have already seen an example: the "limit" of twisted cubics from last time. Where the branches intersected there were elements like z^2.

If X is a complex space we obtain in the usual way a local ring $\mathcal{O}_{X,x}$, which has a maximal ideal $\mathfrak{m}_x$ of functions vanishing at x. The main example for us at the moment is the "double point." We write

$$X = \operatorname{Spec} \mathbb{C}[\varepsilon]$$

for the complex space that is topologically just a point $* \hookrightarrow \mathbb{C}$ but whose ring of functions is $\mathcal{O}_X = \mathbb{C}[\![T]\!]/(T^2) = \mathbb{C}[T]/(T^2)$.

Definition 323. The tangent space of a complex space X at x is by definition

$$T_xX := \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^\vee = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C}).$$

Remark 324. If you've never seen this before, you might think this is a pretty weird definition. But it's just a definition that works in great generality. For usual manifolds X we think of the tangent space at a point x as derivations of functions in $\mathcal{O}_{X,x}$. Of course, in coordinates this is just objects of the form $\sum_i a_i\partial_i$. In general,

$$\operatorname{Der}(\mathcal{O}_{X,x}) \cong \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C}).$$

For instance, given D a derivation we notice that $D(\mathfrak{m}_x^2) = 0$ by the Leibniz rule whence we obtain a map on the quotient. On the other hand given θ on the right we just define $D(f) = \theta(\overline{f - f(x)})$ which one checks is a derivation.

Definition 325. A **morphism of complex spaces** is a map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consisting of the data of a continuous map $X \to Y$ as well as a morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$. The **fiber** of $y \in Y$ is the complex space $(f^{-1}(y), \mathcal{O}_X/f^{-1}\mathfrak{m}_y)$.

Notice for instance that the map $z \mapsto z^2$ on $\mathbb{C} \to \mathbb{C}$ as complex manifolds, viewed as a map of complex spaces, has the fiber the double point at 0.

We now replace the submersion condition

Definition 326. We say that f a map of complex spaces is **flat** if for all $x \in X$, the ring homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat map of rings. Recall that this means that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module, i.e. tensoring with $\mathcal{O}_{X,x}$ is an exact functor on the category of $\mathcal{O}_{Y,f(x)}$-modules.

We are interested now in families where the total space and base may not be manifolds but the fibers are still manifolds. For instance we might have a family over $\mathbb{C}^2$ and want to restrict it to a node or something singular in the plane.

Definition 327. More generally, a smooth family of complex manifolds is a flat proper morphism of complex spaces $\pi : \mathcal{X} \to B$ such that all fibers $X_t = f^{-1}(t)$ are compact complex manifolds. We will also refer to $\mathcal{X}$ as a deformation of some fiber X_0.

Definition 328. An **infinitesimal deformation** (of the first order) of a compact complex manifold X is a smooth family $\pi : \mathcal{X} \to \operatorname{Spec} \mathbb{C}[\varepsilon]$ such that there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \operatorname{Spec} \mathbb{C}[\varepsilon] \end{array}$$

We usually say that the fiber of π over the "closed point" 0 is X.

Recall the ring of functions on $\operatorname{Spec} \mathbb{C}[\varepsilon]$ is $\mathbb{C}[T]/(T^2)$ whence we have a short exact sequence (coming from functions of the bottom row of the diagram)

$$0 \to (T)/(T^2) \to \mathbb{C}[T]/(T^2) \to \mathbb{C}[T]/(T) \to 0.$$

Proposition 329. *Let X be a compact complex manifold. Then there is a one-to-one correspondence between infinitesimal deformations of X (up to isomorphism) and the cohomology group $H^1(X; T_X)$.*

We interpret this as: these sets should form the tangent space to a space of deformations.

Lemma 330. *If B is a complex space then a morphism of complex spaces* $\operatorname{Spec} \mathbb{C}[\varepsilon] \to B$ *is precisely the data of a point $t \in B$ and a tangent vector $v \in T_t V$.*

45. May 11, 2018

Lemma 331. *Let B be a complex space. Then a morphism* $\operatorname{Spec}\mathbb{C}[\varepsilon] \to B$ *is precisely the data of a point $t \in B$ and a tangent vector $v \in T_t B$.*

Proof. The underlying map of topological spaces fixes a point t. We have additionally a map of sheaves $\mathcal{O}_B \to f_*\mathcal{O}_{\operatorname{Spec}\mathbb{C}[\varepsilon]}$. In other words, for each open set $U \subset B$ there is a $\mathbb{C}$-algebra homomoprhism $\mathcal{O}_B(U) \to \mathbb{C}[T]/(T^2)$. If $T \notin U$ then this map is of course the zero map. Otherwise, we see that the map is completely determined by its behavior on the stalk at t. Hence it is the data of a $\mathbb{C}$-algebra map $\phi : \mathcal{O}_{B,t} \to \mathbb{C}[T]/(T^2)$. Such a map must send $\mathfrak{m}_t \mapsto (T)$ whence $\phi(\mathfrak{m}_t^2) = 0$ so we have an element of $(\mathfrak{m}_t/\mathfrak{m}_t^2)^*$. □

Remark 332. If $\pi : \mathcal{X} \to B$ is a smooth family of compact complex manifolds, the data of a point and tangent vector on B, we can take the pullback of the family to $\operatorname{Spec}\mathbb{C}[\varepsilon]$. This is how we associate to a family an infinitesimal deformation (notice that flatness is preserved under base change).

Recall last time we had the Kodaira-Spencer map, which was a map $T_0 B \to H^1(X; T_X)$.

Proposition 333. *If X is a compact complex manifold then there exists a one-to-one correspondence between infinitesimal deformations of X up to isomorphism and $H^1(X; T_X)$.*

Proof. Let $\mathcal{X}$ be an infinitesimal deformation. Choose an open cover $\{U_i\}_{i \in I}$ with $U_i \cong B_i \in \mathbb{C}^n$ of X. Write $U_{ij} = U_i \cap U_j$. First notice that $\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[T]/(T^2)$. In particular $\mathcal{O}_{\mathcal{X}}|_{U_i} = \mathcal{O}_{U_i} \otimes_{\mathbb{C}} \mathbb{C}[T]/(T^2)$. We now have trivializations

$$\theta_i : \mathcal{O}_{\mathcal{X}}|_{U_i} \to \mathcal{O}_{U_i}[T]/(T^2).$$

and transition functions

$$\theta_{ij} : \mathcal{O}_{U_{ij}}[T]/(T^2) \to \mathcal{O}_{U_{ij}}[T]/(T^2)$$

where $\theta_{ij} = \theta_i \circ \theta_j^{-1}$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T/(T^2) & \longrightarrow & \mathcal{O}_{U_{ij}}[T]/(T^2) & \longrightarrow & \mathcal{O}_{U_{ij}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta_{ij}} & & \downarrow & & \\ 0 & \longrightarrow & T/(T^2) & \longrightarrow & \mathcal{O}_{U_{ij}}[T]/(T^2) & \longrightarrow & \mathcal{O}_{U_{ij}} & \longrightarrow & 0 \end{array}$$

We have that

$$\theta_{ij}(f) = f + T\psi_{ij}(f)$$

for some $\psi_{ij}(f)$ since θ_{ij} has to be the identity on $\mathcal{O}_{U_{ij}}$. Since θ_i has to be a ring homomorphism we see that

$$fg + T\psi_{ij}(fg) = (f + T\psi_{ij}(f))(g + T\psi_{ij}(g)) = fg + T(f\psi_{ij}(g) + g\psi_{ij}(f))$$

so ψ_{ij} must be a derivation of $\mathcal{O}_{U_{ij}}$. In other words,

$$\psi_{ij} \in \Gamma(U_{ij}, T_X).$$

The θ_{ij} glue if and only if they satisfy the cocycle condition

$$\theta_{ij}\theta_{jk}\theta_{ki} = \mathrm{id},$$

whence (using additive notation for ψ)

$$\psi_{ij} + \psi_{jk} + \psi_{ki} = 0$$

on U_{ijk} and we obtain $[\psi_{ij}] \in \check{H}^1(\mathcal{U}; T_X)$. Clearly this map, from infinitesimal deformations modulo isomorphisms to these cohomology classes, is surjective, using our previous lemma.

To check well-definedness and injectivity, pick two trivializations θ_i and θ_i'. They give the same infinitesimal deformation $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\varepsilon]$ if and only if they differ by composition with some $\mu_i : \mathcal{O}_{U_i}[T]/(T^2) \to \mathcal{O}_{U_i}[T]/(T^2)$ where $\mu_i = \mathrm{id}$ on U_i and $\mu_i(T) = T$ (exercise). But as before $\mu_i(f) = f + T\psi_i(f)$ with ψ_i a vector field on U_i. A straightforward calculation reveals that

$$\psi'_{ij}\psi_{ij} + \psi_i - \psi_j.$$

But this exactly means that $[\psi_{ij}] = [\psi'_{ij}]$ in $\check{C}^1(\mathcal{U}, T_X)$. Now take a limit over all refinements to obtain a class $\eta \in H^1(X, T_X)$. □

We now turn to a slightly different, more analytic, point of view. Consider a fixed manifold instead of a family and look at the space of complex structures on it. More precisely, let X be a smooth manifold. An almost complex structure on X is an endomorphism $J : TX \to TX$ of the real tangent bundle such that $J^2 = -\mathrm{id}$. The data of an almost complex structure is equivalent to the data of a decomposition

$$TX \otimes_{\mathbb{R}} \mathbb{C} = T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

In particular

$$T^{1,0}X = \ker(J - i\,\mathrm{id}) \qquad T^{0,1}X = \ker(J + i\,\mathrm{id}).$$

If X is in fact a complex manifold, then the $T^{1,0} \cong T_X$ is the holomorphic tangent bundle of X.

Theorem 334 (Newlander-Nirenberg)**.** *An almost complex structure comes from the structure of a complex manifold on X (i.e. is "integrable") if and only if*

$$[T^{0,1}X, T^{0,1}X] \subset [T^{0,1}X, T^{0,1}X].$$

Suppose we have a one-parameter family $J(t)$ of almost complex structures. We will have a varying decomposition

$$T_{\mathbb{C}}X = T_t^{1,0} \oplus T_t^{0,1},$$

which we will study closely in order to obtain, in this perspective, the result from before.

46. May 14, 2018

Let (X, J) be a complex manifold and $J(t)$ be a smooth one-parameter family of almost complex structures for $t \in T$ some parameter space, such that $J(0) = J$. We have a decomposition for each $t \in T$,

$$T_{\mathbb{C}}X = T^{1,0}X_t \oplus T_t^{0,1}.$$

Consider the composition

$$T^{0,1} \hookrightarrow T_{\mathbb{C}}X \to T_t^{1,0}$$

of the inclusion at time zero and the projection and similarly the composition

$$T^{1,0} \hookrightarrow T_{\mathbb{C}}X \to T_t^{1,0}.$$

Since the latter is at $t = 0$ the identity, we see that it is an isomorphism for all t in some small neighborhood of $0 \in T$. Thus for small t we obtain a linear transformation

$$\phi_t : T^{0,1} \to T^{1,0}$$

by including at time zero and projection and then applying the inverse of the isomorphism mentioned above. For $v \in T^{0,1}$ we have that $v - \phi_t(v) \in T_t^{0,1}$. Conversely, given $\phi_t : T^{0,1} \to T^{1,0}$ we can define $T_t^{0,1} = \operatorname{im}(\operatorname{id} + \phi_t)$. In other words, a one-parameter family of almost complex structures in the neighborhood of 0 is the same as a family of transformations $\phi_t : T^{0,1} \to T^{1,0}$. The latter is just a section in $\Gamma(X, \mathcal{A}^{0,1}(T^{1,0}))$. We are interested in expanding

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$$

which each ϕ_i again a section. Since at $t = 0$ we should have no deformation we see that $\phi_0 = 0$.

Question: When does $J(t)$ correspond to a complex structure on X? Recall that the Newlander-Nirenberg theorem tells us that $J(t)$ is a complex structure if and only if $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$. Recall that we have a $\bar\partial$ operator

$$\bar\partial : \mathcal{A}^{0,p} T_X \to \mathcal{A}^{0,p+1} T_X.$$

Moreover we have a combination of wedging and Lie bracketing

$$[-,-] : \mathcal{A}^{0,p} T_X \otimes \mathcal{A}^{0,q} T_X \to \mathcal{A}^{0,p+q} T_X,$$

which sends for instance $[v_1 \otimes w_1, v_2 \otimes w_2] = v_1 \wedge v_2 \otimes [w_1, w_2]$. Locally in coordinates $z_1, \ldots, z_n$ this is given by

$$[\sum_{I,j} f_I d\bar z_I \otimes \partial_{z_j}, \sum_{J,l} g_{Jl} d\bar z_j \otimes \partial_{z_l}] = \sum_{I,J,j,l} (d\bar z_I \wedge d\bar z_j [f_{IJ} \partial_{z_j}, g_{Jl} \partial_{z_l}].$$

Proposition 335. *The integrability condition of Newlander-Nirenberg is equivalent to the **Maurer-Cartan equation***

$$\bar\partial \phi_t + [\phi_t, \phi_t] = 0.$$

Proof. Consider the problem for each t for the moment. Omitting it from the notation we have

$$\phi = \sum_{i,j} \phi_{ij} d\bar z_i \otimes \partial_{z_j}.$$

Recall $T_t^{0,1} = (\operatorname{id} + \phi_t)(T^{0,1})$. Assume that the Newlander-Nirenberg condition holds. For each i, k, we have

$$[\partial_{\bar z_i} + \phi(\partial_{\bar z_i}), \partial_{\bar z_k} + \phi(\partial_{\bar z_k})] \in T_t^{0,1}.$$

Suppose that $i \neq k$. Then

$$[\partial_{\bar z_i}, \partial_{\bar z_k}] = 0$$

and

$$\phi(\partial_{\bar z_i}) = \sum_j \phi_{ij} \partial_{\bar z_j}.$$

The big commutator above now becomes

$$\sum_l [\partial_{\bar z_i}, \phi_{kl} \partial_{z_l}] + \sum_j [\phi_{ij} \partial_{z_j}, \partial_{\bar z_k}] + \sum_{jl} [\phi_{ij} \partial_{z_j}, \phi_{kl} \partial_{z_l}].$$

The first two terms, it is easy to check, yield

$$\sum_l \frac{\partial \phi_{kl}}{\partial \bar{z}_i}\partial_{z_l} - \sum_j \frac{\partial \phi_{ij}}{\partial \bar{z}_k}\partial_{z_j} = \sum_j \left(\frac{\partial \phi_{kj}}{\partial \bar{z}_i} - \frac{\partial \phi_{ij}}{\partial \bar{z}_k}\partial_{z_j} \right)$$

but this is precisely

$$\bar{\partial}\phi(\partial_{\bar{z}_i}, \partial_{\bar{z}_k}).$$

Finally by the definition of the bracket we have

$$[\phi, \phi] = \sum_{ijkl} (d\bar{z}_i \wedge d\bar{z}_k)[\phi_{ij}\partial_{z_j}, \phi_{kl}\partial_{z_l}].$$

Applying this to the pair $\partial_{\bar{z}_i}, \partial_{\bar{z}_k}$ we obtain precisely the third term above. We conclude that

$$(\bar{\partial}\phi + [\phi, \phi])(\partial_{\bar{z}_i}, \partial_{\bar{z}_k}) \in T_t^{0,1}.$$

Hence $\bar{\partial}\phi + [\phi, \phi] \in \mathcal{A}^{0,2}(T_X \cap T_t^{0,1})$ for t small but $T_X \cap T_t^{0,1} = T^{1,0} \cap T_t^{0,1} = 0$ for t small. We conclude that $\bar{\partial}\phi + [\phi, \phi] = 0$.

The converse is straightforward. □

Now suppose we expand ϕ (formally for now) in a power series in t. The Maurer-Cartan equation now becomes a sequence of equations that we can try to solve inductively. In particular we must have that ϕ_1 is $\bar{\partial}$-closed whence $[\phi_1] \in H^1(X, T_X)$ (by the Dolbeault theorem).

47. May 16, 2018

We will have no class this Friday.

Recall from last time we had a Maurer-Cartan equation that was equivalent to the integrability of an almost complex structure. In particular we had sections $\phi_i \in \Gamma(X, \mathcal{A}^{0,1}T_X)$ and we wrote formally

$$\phi(t) = \phi_i t + \phi_2 t^2 + \cdots .$$

If we insert this into the Maurer-Cartan equation we obtain a family of equations which we might try to solve inductively. The first equation is given $\bar{\partial}\phi_1 = 0$ whence a solution is $[\phi_1] \in H^{0,1}(T_X) \cong H^1(X, T_X)$. Such a solution class is often called a **first order deformation** or **infinitsimal deformation**. The solution for the ϕ_i we would thus call an ith order deformation.

The following result follows from the fact that first-order deformations are isomorphic if and only if they differ, roughly, by conjugation by a diffeomorphism.

Proposition 336. *Isomorphism classes of first-order deformations are in bijection with $H^1(X, T_X)$.*

We now want to "integrate" first-order deformations, i.e. given a class in $H^1(X, T_X)$ find a one-parameter family of complex structures $J(y)$ such that the associated $[\phi_1]$ equals our fixed class. There is however no a priori reason we should be able to solve these equations. Consider the following immediate obstruction: look at the second equation $\bar{\partial}\phi_2 + [\phi_1, \phi_1] = 0$. It is an exercise to show that if $\alpha \in \mathcal{A}^{0,p}T_X, b \in \mathcal{A}^{0,q}T_X$ then

$$\bar{\partial}[a, b] = [\bar{\partial}a, b] + (-1)^p [a, \bar{\partial}b].$$

It follows that $\bar{\partial}[\phi_1, \phi_1] = 0$, i.e. there is a class $[\phi_1, \phi_1] \in H^2(X, T_X)$. Solving the second equation means, then, that this class needs to be exact, witnessed by ϕ_2. Notice that there is a cup product map

$$H^1(X, T_X) \times H^1(X, T_X) \to H^2(X, T_X)$$

sending $(v, v) \mapsto [v, v]$ so really we are looking at the image of our first order deformation under this map. For this reason we often call $H^2(X, T_X)$ the obstruction space.

In fact, if we look at the higher pieces of the Maurer-Cartan equation we still are looking at commutators of $(0, 1)$-forms so actually this second cohomology is relevant for *all* the "liftings" from lower-order to higher-order deformations.

It turns out that formally this is actually an if and only if statement.

Proposition 337. *If $H^2(X, T_X) = 0$ then any $v \in H^1(X, T_X)$ can be formally integrated.*

Notice that if X is a curve we're good to go for dimensional reasons: any first-order deformation yields a full deformation. Another example is for K3 surfaces, where by Serre duality $H^2(X, T_X) \cong H^0(X, \Omega^1_X) = 0$ where we have used that ω_X is trivial. Moreover in this case $H^1(X, T_X) \cong H^1(X, \Omega^1_X)$ which is 20-dimensional as we discussed before. If we look at Calabi-Yau manifolds in general it gets a little trickier. We can still write

$$H^2(X, T_X) \cong H^{n-2}(X, \Omega^1_X)^\vee \cong (H^{1,n-2}X)^\vee.$$

One might ask when this vanishes – notice there is a problem for threefolds immediately because Calabi-Yaus of dimension greater than 2 are projective and we will have $H^{1,1}X \neq 0$! However it turns out that Calabi-Yaus are nontheless unobstructed, for much deeper reasons. This is known as the Tian-Todorov theorem.

Proof. Fix $v \in H^1(X, T_X)$. Our above computation shows that we can solve for ϕ_2. Assume inductively that we have extended to $\phi_1 t + \phi_2 t^2 + \cdots + \phi_{k-1}t^{k-1}$. We would like to find a ϕ_k such that

$$\bar\partial \phi_k = -\sum_{0<i<k} [\phi_i, \phi_{k-i}] =: \omega_{k-1}.$$

Since $H^2(X, T_X) = 0$ it is enough to show that $\bar\partial\omega_{k-1} = 0$.

We compute, using inductively our previous solutions,

$$\begin{aligned}\bar\partial\omega_{k-1} &= -\sum_{0<i<k} \bar\partial[\phi_i, \phi_{k-i}] \\ &= \sum_{0<i<k} [\bar\partial\phi_i, \phi_{k-i}] - [\phi_i, \bar\partial\phi_{k-i}] \\ &= \sum_{i+j=k, i,j>0} [\bar\partial\phi_i, \phi_j] - [\phi_i, \bar\partial\phi_j] \\ &= \sum_{p+q+r=k, p,q,r>0} [[\phi_p, \phi_q], \phi_r] - [\phi_p, [\phi_q, \phi_r]].\end{aligned}$$

Now we point out that the bracket $[-,-]$ on $\mathcal{A}^{0,1}T_X$ satisfies

$$[a, b] = [b, a], \qquad [a, [b, c]] = [[a, b], c] - [b, [a, c]]$$

For a discussion for these algebra structures see either Huybrechts or anything by Marco Manetti. We now have that

$$\bar\partial\omega_{k-1} = \sum_{p+q+r=k, p,q,r>0} [\phi_q, [\phi_q, \phi_r]].$$

We now leave it as an exercise to show that this vanishes – simply match like terms using the algebraic identities above. □

So far we have a "formal solution". But we are interested in constructing a family acheiving this deformation, geometrically.

Definition 338. We say that $\pi : \mathcal{X} \to B \ni 0$ is a **complete deformation** of X if any other such family $\pi : \mathcal{X}' \to B' \ni 0'$ is obtained as a pullback

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

where the map in the bottom row sends $0' \mapsto 0$. If this map f is unique then we say that this complete deformation is moreover **universal**. If only $df_{0'}$ is unique then we call this deformation **versal**.

Kodaira's book on deformation theory (or the paper of Kodaira and Spencer) is devoted to the following result.

Theorem 339 (Kodaira-Spencer)**.** *If B is reduced then $\pi : \mathcal{X} \to B$ is complete if and only if the Kodaira-Spencer map*

$$\kappa : T_0 B \to H^1(X, T_X)$$

is surjective.

Theorem 340 (Kuranishi)**.** *Any compact complex manifold admits a versal deformation.*

Here are some stronger results.

- If $H^0(X, T_X) = 0$ then any versal deformation is universal. We think of this cohomology group as the tangent space to the connected component of the identity of the automorphism group of the manifold.
- If $H^2(X, T_X) = 0$ then X admits a smooth versal deformation (people say that X is "unobstructed"). What we saw earlier was the infinitesimal, tangent space version of this.

That ends our discussion of deformation theory. We will return to Hodge structures and their variation.

48. May 21, 2018

48.1. **Local systems.** Let B be a topological space (later the base space of a family of compact complex manifolds). Assume that B is connected and (locally) path-connected. What we will discuss first is a toy version of the Riemann-Hilbert correspondence that is due to Deligne. In particular, there is an equivalence between the categories of local systems of $\mathbb{C}$-vector spaces on B, representations $\rho : \pi_1(B) \to GL_n(\mathbb{C})$ for arbitrary n, and vector bundles with flat connection on B.

Definition 341. A **local system** on B is a locally constant sheaf $\mathcal{L}$ on B with fiber some vector space $V \cong \mathbb{C}^n$.

Recall that locally constant means that there exists an open cover $\{U_i\}$ of B such that $\mathcal{L}|_{U_i}$ is the constant sheaf V. Notice that the transition functions are constant (on each connected component).

Example 342. Let $\pi : \mathcal{X} \to B$ be a smooth family of compact complex manifolds. Let $\mathbb{C}$ be the constant sheaf on X with fiber $\mathbb{C}$. Say that $\mathcal{F}$ is any sheaf on $\mathcal{X}$. Then we obtain a sheaf $\pi_*\mathcal{F}$ which assigns to $U \subset B$ the space $H^0(f^{-1}(U), \mathcal{F})$. In fact we have the higher direct images $R^k\pi_*\mathcal{F}$, which is the sheaf associated to the presheaf

$$U \mapsto H^k(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)}).$$

If we fix a point $0 \in B$ then for any contractible neighborhood U of $0 \in B$ we know that $\pi^{-1}(U)$ is (C^∞-) diffeomorphic to $U \times X_0$. The induced map $\pi^{-1}(U) \to X_0$ is just a deformation retraction, from which it follows $H^k(\pi^{-1}(U), \mathbb{C}) \cong H^k(X_0, \mathbb{C})$. Hence if we consider $R^k\pi_*\mathbb{C}$ we obtain a constant sheaf on U with fiber $H^k(X, \mathbb{C})$, whence $R^k\pi_*\mathbb{C}$ is a local system. The stalk of $R^k\pi_*\mathbb{C}$ at some point $t \in B$ is canonically $H^k(X_t, \mathbb{C})$.

Example 343. Let $U \subset \mathbb{C}^n$ with coordinates z. Consider a holomorphic map $U \to GL_n(\mathbb{C})$ and consider the first-order holomorphic system of linear differential equations

$$\frac{du}{dz} = A(z)u.$$

Say $\gamma : [0, 1] \to \mathbb{C}^n$ is an path. Cauchy's theorem (really Cartan's analog in the holomorphic setting) tells us that local solutions with initial conditions at $\gamma(0)$ can be extended along $\gamma([0, 1])$. Over a small open set U the solutions form a vector space and in fact a local system over U.

Lemma 344. *A local system $\mathcal{L}$ on $[0, 1]$ is constant.*

Proof. Cover $[0, 1]$ with open intervals I_t around $t \in [0, 1]$ such that $\mathcal{L}|_{I_t}$ is constant. Inside the open interval I_t let J_t be the middle third. Choose an integer $n \in \mathbb{N}$ such that the length of J_t is greater than $1/n$ for all t. Hence $[p/n, (p + 1)/n]$ is contained in some I_t for every p. Hence $\mathcal{L}$ is constant on each of the $[p/n, (p+1)/n]$. It immediately follows that $\mathcal{L}$ is constant. □

Proposition 345. *Say B is simply connected (and connected and path-connected). Then every local system $\mathcal{L}$ on B is constant.*

Proof. Fix $x \in B$. We wish to show that every element $s_x \in \mathcal{L}_x$ can be extended uniquely to a section $s \in \Gamma(X, \mathcal{L})$ along the restriction map

$$\Gamma(X, \mathcal{L}) \to \mathcal{L}_x.$$

Take any $y \in B$ and any path $\gamma : [0,1] \to B$ from x to y. Choose an open neighborhood U_y of y such that $\mathcal{L}|_{U_y}$ is constant. The previous lemma showed that $\mathcal{L}|_{\gamma([0,1])}$ is constant. Hence there is a unique "transport" $s_y \in \mathcal{L}_y$ of s_x. Next there is a unique extension of s_y to $s_i \in \Gamma(U_y, \mathcal{L}|_{U_y})$. Now take y_1, y_2 such that $U_{y_1} \cap U_{y_2} \neq \varnothing$. We need to show that $(s_{y_1})_z = (s_{y_2})_z$ for the extension to be well-defined. Notice that we get two different paths from x to z, say γ_1 and γ_2. These paths are, by assumption, homotopic. Now we carry out the same argument as we did for the interval but now using the square to conclude that the local system is constant along the image of $[0,1] \times [0,1]$. The result follows. □

Now fix a point $b_0 \in B$ and take any other point $b_1 \in B$. Let $\gamma : [0,1] \to B$ be a path from b_0 to b_1. The restriction $\gamma^*\mathcal{L}$ of any local system is constant. Therefore ev_{b_0} and ev_{b_1} map into the same vector space V (with source $\Gamma([0,1], \gamma^*\mathcal{L})$). Both of these maps are isomorphisms (since the local system is constant) and hence we obtain an automorphism of V. If $b_0 = b_1$ then we obtain a loop at b_0 and thus, using the construction just described, we find a representation of the fundamental group

$$\rho : \pi_1(B, b_0) \to GL(V).$$

This is known as the monodromy representation.

Next time we will reverse this construction.

49. May 23, 2018

49.1. **Monodromy representation.** Let $\mathcal{L}$ be a local system with fiber B on B connected and path connected with basepoint b_0. Last time we constructed a representation $\rho : \pi_1(B, b_0) \to GL(V)$. It is independent up to conjugation of the choice of basepoint.

Proposition 346. *There is a one-to-one correspondence (in fact an equivalence of categories) between local systems on B of rank n (with fiber V) and representations $\pi_1(B, b_0) \to GL_n(V)$.*

Proof. We have written down a construction in one direction. The idea for the other direction is straightforward: the universal cover $p : \tilde{B} \to B$ is equipped with a transitive action of $\pi_1(B, b_0)$ so we take a trivial localy system on the cover and define our local system to be the equivariant subsheaf. Notice that we can write

$$\Gamma(U, \mathcal{L}) = \{s \in \Gamma(p^{-1}(U), V) \mid \sigma(s) = \rho(\sigma)s, \forall \sigma \in \pi_1(B, b_0)\}.$$

Notice that we could do the following. Define

$$\mathcal{V} = \tilde{B} \times V / \sim$$

where $(x, v) \sim (\sigma(x), \rho(\sigma^{-1})v)$. This has a projection map $\pi : \mathcal{V} \to B$ given $(x, v) \to p(x)$. This is the vector bundle associated to the covering space thought of as a principal bundle. Notice that the transition functions are constant! We take $\mathcal{L}$ to be the subsheaf of $\mathcal{V}$ generated by the equivariant sections. □

Example 347. Write $B = \Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. We have that $\pi_1(\Delta^*, 1/2) \cong \mathbb{Z}$. Let γ_0 be a generator represented by a loop going clockwise around 0. Let $\rho : \pi_1(\Delta^*) \to GL_2(\mathbb{C})$ be given

$$k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

The universal cover is the upper half plane $p : \mathbb{H} \to \Delta^*$ sending $z \mapsto \exp(2\pi i z)$. The vector bundle we obtain is

$$\mathcal{V} = \mathbb{H} \times \mathbb{C}^2 / \sim \to \Delta^*.$$

What does a typical section look like? On a small neighborhood of a point we have constant sections, $v \in \mathbb{C}^2$ with $\hat{v} : U \to \mathcal{V}$, $x \to [(p_j^{-1}(x), v)]$. Consider the nilpotent matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and notice that γ_0 is sent to $\mathrm{id} + N$ under our representation whence, since $N^2 = 0$, we have that $\rho(\gamma_0^{-1}) = e^{-N} = \mathrm{id} - N$. For $v \in \mathbb{C}^2$ define $\tilde{v} : \Delta^* \to \mathcal{V}$ to send $w \mapsto [(\log(w)/2\pi i, \exp(\log(w)N/2\pi i \cdot v))]$. We leave it as an exercise to check that this is indeed a section of $\mathcal{V}$.

If we are on a U simply connected, we can choose unambiguously a branch of the logarithm and write

$$\tilde{v} = \exp(\log(w)/2\pi i N)\hat{v}.$$

This is what we think of as a (covariantly) constant section.

49.2. **Vector bundles with (flat) connection.** Let $E \to B$ be a holomorphic vector bundle on B a complex manifold.

Definition 348. A **holomorphic connection** ∇ on E is a map of sheaves

$$\nabla : E \to \Omega_B^1 \otimes E$$

that is $\mathbb{C}$-linear and satisfyies the Leibniz rule (with repsect to $\mathcal{O}_X$).

Notice that this is a stronger notion than our previous smooth definition. In fact, some vector bundles do not admit holomorphic connections. Similarly as before we have that a connection is determined locally by a matrix of holomorphic one-forms. Choose a local frame $s_1, \ldots, s_r$ over $U \subset B$ and write

$$\nabla s_j = \sum_i \theta_{ij} \otimes s_i.$$

Remark 349. If we fix $\xi \in \Gamma(X, T_X)$ we write, locally,

$$\nabla_\xi s = \sum_{i=1}^{r} \left(\xi(f_i) + \sum_j f_j \theta_{ij}(s_j) \right) s_i.$$

Next time we will discuss curvature and in particular flat connections.

50. May 25, 2018

Let B be a complex manifold and $E \to B$ a holomorphic vector bundle on B. Let $\nabla : E \to E \otimes \Omega_B^1$ be a holomorphic connection. Fix a local frame $s_1, \dots, s_r \in \Gamma(U, E)$, with respect to which we can write

$$\nabla s_j = \sum_{i=1}^{r} \theta_{ij} s_i$$

where (θ_{ij}) is a matrix of holomorphic one-forms.

Remark 350. If $s_1', \dots, s_r'$ is another frame with $s_i' = \sum g_{ij} s_i$ with $g_{ij} \in \mathcal{O}_B(U)$ forming an invertible matrix and associated matrix of one-forms (θ'_{ij}). Then it is easy to check that

$$\theta' = g^{-1} dg + g^{-1} \theta g.$$

Consider now the composition $\nabla \circ \nabla : \Omega_B^0 \otimes E \to \Omega_B^2 \otimes E$ as usual. We call this the curvature of ∇, which we consider as a section in $\Gamma(X, \Omega_B^2 \otimes \operatorname{End} E)$. Locally, if we fix a frame, we have that

$$\nabla^2 s_j = \sum_{i=1}^{r} \Theta_{ij} s_i$$

where, as in the smooth setting,

$$\Theta_{ij} = d\theta_{ij} - \sum_{k=1}^{r} \theta_{ik} \wedge \theta_{kj}$$

or in matrix notation

$$\Theta = d\theta - \theta \wedge \theta.$$

Remark 351. Carrying out a change of frame one finds a simpler formula for the curvature $\Theta' = g^{-1} \Theta g$.

Definition 352. A section of E is **flat** or **covariantly constant** if $\nabla s = 0$. We say that ∇ is **flat** or **integrable** if there exists a trivializing open cover $\{U_i\}$ together with frames $\{s_i^1, \dots, s_i^r\}$ for $\Gamma(U_i, E)$ such that all s_j^k are flat sections.

Proposition 353. *∇ is flat if and only if $\nabla^2 = \Theta = 0$.*

Proof. This is basically the Frobenius theorem in linear PDE. See, for instance, Voisin's book or Kobayashi/Nomizu. □

Proposition 354. *There is a one-to-one correspondence (really an equivalence of categories) between vector bundles of rank r with flat connection and local systems of rank r.*

Proof. Let $(\mathcal{V}, \nabla)$ be a bundle with flat connection. Consider the sheaf of flat sections $\mathcal{L}(U) = \{s \in \Gamma(U, \mathcal{V}) \mid \nabla s = 0\}$. We claim that this is a local system of rank r. Let's sketch how the flatness condition yields differential equations. Let $U \subset B$ be an open set such that we have a trivialization $\phi : \mathcal{V}|_U \to \mathcal{O}_U^{\oplus r}$. There is a local frame corresponding to the standard basis on the trivial bundle. For each $s \in \Gamma(U, \mathcal{V})$ we have $s = \sum_{j=1}^{r} f_j s_j$ for $f_j \in \mathcal{O}_X(U)$. We compute

$$\nabla(f_j s_j) = df_j \otimes s_j + f_j \nabla s_j.$$

But recall that $\nabla s_j = \sum_i \theta_{ij} s_i$ so, if we work in $\mathcal{O}_U^{\oplus r}$

$$\nabla s = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \theta \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Now say $z_1, \ldots, z_n$ are local coordinates on $\mathbb{C}^n$. Then since

$$\theta_{ij} = \sum \Gamma_{ij}^k(z) dz_k$$

the condition that $\nabla s = 0$ is equivalent to

$$\frac{\partial f_i}{\partial z_k} + \sum_{j=1}^r \Gamma_{ij}^k f_j = 0$$

for each $i = 1, \ldots, r$ and $k = 1, \ldots, n$. This is now a linear system of PDEs whose solutions, by the theorem, above form a local system.

Now say $\mathcal{L}$ is a local system of rank r. We constructed, last time, a bundle $\mathcal{V} = \mathcal{L} \otimes \otimes_{\mathbb{C}} \mathcal{O}_X$ of rank r with constant transition functions coming from $\mathcal{L}$. Define a connection $\nabla : \mathcal{V} \to \Omega_B^1 \otimes \mathcal{V}$ as follows. Every section can be decomposed $s = \sum f_i s_i$ for s_i a constant frame. Then define

$$\nabla(\sum f_i s_i) = \sum df_i \otimes s_i,$$

i.e. the connection where $\theta_{ij} = 0$. It follows immediately, from the formula, say, that $\Theta = 0$. □

50.1. **Back to families.** Let $\pi : \mathcal{X} \to B$ of compact complex manifolds, with $\mathcal{X}$ and B smooth. Fix $b \in B$ and write $X = X_0$. Fix $k \in \mathbb{N}$. Consider $V = H^k(X, \mathbb{C}) \cong \mathbb{C}^r$. To this data we associated a local system

$$\mathcal{L} = R^k \pi_* \mathbb{C}.$$

Equivalently this is the data of a representation $\rho : \pi_1(B, 0) \to GL(V)$ or the data of a bundle with flat connection $(\mathcal{V} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X, \nabla)$ where $\nabla = \nabla_{\mathsf{GM}}$ is called the **Gauss-Manin connection**.

Remark 355. Notice that the issue of monodromy is generally related to where fibers become singularities. In fact it turns out that when the monodromy is trivial then the central fiber has to be smooth.

We have a **Cartan-Lie** formula for the Gauss-Manin connection. Locally over U a section of $\mathcal{V}$ assigns smooth to $t \in U$ closed forms $\omega_t \in A^k(X_t)$. If U is sufficiently small then $\pi^{-1}(U)$ deformation retratcts to X. Hence we obtain Ω on $A^k(\pi^{-1}(U))$ such that $\Omega|_{X_t} = \omega_t$. Choose local coordinates $t_1, \ldots, t_n$ on the base and consider a lift of the vector field ∂_{t_i} called v

$$\nabla_{\partial_{t_i}} s = [\iota_v(d\Omega|_{X_0})].$$

In the next few classes we will move towards Griffith's transversality.

51. May 30, 2018

We have 3 classes left including today so we will skip some of the proofs. As before consider a smooth family $\pi : \mathcal{X} \to B$ of compact complex manifolds but now assume that the fibers are moreover Kähler. Let $E \to X$ be a holomorphic vector bundle and denote by $E_t = E|_{X_t}$ for $t \in B$.

Theorem 356 (Semicontinuity)**.** *For every $i \geq 0$ the map $\phi : B \to \mathbb{N}$ given*

$$\phi : t \mapsto h^i(X_t, E_t)$$

is upper semicontinuous, i.e. for each t_0 there is a neighborhood of t_0 such that for each t in the neighborhood we have that $\phi(t) \leq \phi(t_0)$.

Proof. See Voisin's book, for instance, though she's using a harder theorem to prove it. . . it's a pretty nontrivial theorem. □

In other words, cohomology can only jump up under specialization. In particular the dimension of cohomology can only go up on proper closed subsets. It turns out that the Kähler condition is crucial, as we can see analytically. We can fix a Hermitian metric for E which yields induced metrics for E_t. For each of these we can define an $\bar{\partial}_t$ operator on forms with values in E_t. We can use Hodge theory (since we are Kähler) and relate the cohomology to the kernel of a family of Laplacians, and then apply a result on semicontinuity of dimensions of solutions.

Remark 357. The theorem in fact works for any coherent sheaf flat over the base, not just vector bundles.

Let's say we have a fiber $F = X_t$ over t. Then we have a normal bundle sequence

$$0 \to T_F \to T_{\mathcal{X}}|_F \to N_{F|X} \to 0.$$

Holomorphically this is not necessarily split. In any case, this sequence globalizes

$$0 \to T_{\mathcal{X}/B} \to T_{\mathcal{X}} \to \pi^* T_B \to 0$$

where the kernel is what we call the relative tangent bundle (it is a bundle since the map we are taking the kernel of is indeed a map of bundles). One can dualize to obtain

$$0 \to \pi^* \Omega_B^1 \to \Omega_{\mathcal{X}}^1 \to \Omega_{\mathcal{X}/B}^1 \to 0$$

where the cokernel we call the relative cotangent bundle. Notice that

$$\Omega_{\mathcal{X}/B}^1|_F \cong \Omega_F^1.$$

Taking exterior powers we have

$$\Omega_{\mathcal{X}/B}^p := \Lambda^p \Omega_{\mathcal{X}/B}^1.$$

and $\Omega_{\mathcal{X}/B}^p|_F = \Omega_F^p$. We obtain the following from the theorem above.

Corollary 358. *The function $t \to h^{p,q}(X_t)$ is upper semicontinuous for each p, q.*

Corollary 359. *In fact, the function $t \to h^{p,q} X_t$ is constant.*

Proof. The family is smoothly locally trival so the Betti numbers are constant in t. Now just apply the Hodge decomposition:

$$b_k(X_t) = \sum_{p+q=k} h^{p,q} X_t \leq \sum_{p+q=k} h^{p,q} X_{t_0} = b_k(X_{t_0}).$$

□

We will use this constancy to linearize our family in the following sense. Fix $k \geq 0$ and consider the usual local system $\mathcal{L} = \mathsf{R}^k\pi_*\mathbb{C}$ on the base. Consider the associated flat vector bundle

$$(\mathcal{V}, \nabla) = (\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_B, \nabla^{\mathsf{GM}}).$$

For each $t \in B$ we have the (decreasing) Hodge filtration

$$F^p H^k(X_t, \mathbb{C}) := \bigoplus_{r+s=k, r \geq p} H^{r,s}(X_t) \subset H^k(X_t, \mathbb{C}).$$

Write

$$f_p = f_p(t) = \dim_{\mathbb{C}} F^p H^k = \sum_{r \geq p} h^{r,s} X_t,$$

which is constant. Take $U \subset B$ contractible and fix $0 \in U$. Fix $X := X_0$. There is an isomorphism (after choosing a retract)

$$H^k(X_t, \mathbb{C}) \cong H^k(\pi^{-1}(U), \mathbb{C}) \cong H^k(X, \mathbb{C}).$$

Call $V := H^k(X, \mathbb{C})$. Now using these isomorphisms we can view these as subspaces of one vector space V. More precisely, we obtain a map

$$\mathcal{P} : U \to \mathrm{Gr}(f_p, V = H^k(X, \mathbb{C})),$$

which is a part of the "period map". Unwinding all the definitions and the fact that cohomology is detected by Laplacians, one finds that this map is smooth. Griffiths showed in fact that this map is holomorphic! In fact he proved a lot more.

Theorem 360 (Griffiths)**.** *The map $\mathcal{P}_p$ is holomorphic.*

Before we discuss this map, let's recall some basics of Grassmannians. Here are some facts to remember:

- $\mathrm{Gr}(k, n)$ is a compact complex manifold of dimension $k(n-k)$. For $k = 1$, of course, we obtain $\mathbb{P}^{n-1}$. It turns out that $\mathrm{Gr}(k, V)$ is actually projective in general. One can specify an easy embedding
$$\mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\Lambda^k V).$$
- There exists a canonical identification
$$T_W \mathrm{Gr}(k, V) \cong \mathrm{Hom}_{\mathbb{C}}(W, V/W).$$
The dimension of the Grassmanian becomes clear in view of this fact.
- Consider the trivial bundle $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}}$. Inside of this bundle there is a holomorphic subbundle, call it S, which is the tautological vector bundle (think similarly to projective space). Denote by Q the cokernel of this inclusion. It follows that $T_{\mathrm{Gr}} \cong \mathrm{Hom}(S, Q)$.
- Fix $W \subset V$. We want a chart near W of the Grassmannian. Fix a W' such that $V = W \oplus W'$. Write
$$U_{W'} = \{U \in \mathrm{Gr}(k, V) \mid U \cap W' = \{0\}\}.$$
We claim that
$$U_{W'} \cong \mathrm{Hom}_{\mathbb{C}}(W, W') \cong \mathrm{Hom}_{\mathbb{C}}(W, V/W) \cong \mathbb{C}^{k(n-k)}.$$
Intuitively the idea is that all these independent directions can be viewed as graphs.

52. June 1, 2018

Let $W \subset V$. Recall that we claimed last time that $T_W \operatorname{Gr} = \operatorname{Hom}_{\mathbb{C}}(W, V/W)$. Let us sketch why this is true. Consider a one-parameter family of subspaces $W(t)$ of V such that $W(0) = W$. Moreover fix $v \in W$ and consider a one-parameter family of vectors $v(t) \in W(t)$ such that $v(0) = v$. Consider

$$\phi(v) = \frac{d}{dt}|_{t=0} v(t) \in V.$$

Does this depend on the one-parameter family that we chose? If we took another, $w(t)$ with $w(0) = v$ then

$$w(t) - v(t) = tu(t)$$

for some $u(t) \in W(t)$, whence

$$w'(t) - v'(t) = tu'(t) + u(t)$$

from which we find that $w'(0) - v'(0) = u(0) \in W$. Hence this v is unique only if we consider it up to vectors in W.

We now return to the period map. We had a family $\pi : \mathcal{X} \to B$ and chose a contractible neighborhood U of B. Recall the flat vector bundle $(\mathcal{V}, \nabla^{\mathsf{GM}})$. Then we defined, for $0 \le p \le k$

$$\mathcal{P}_p : U \to \operatorname{Gr}(f_p, H^k(X, \mathbb{C}))$$

sending $t \mapsto F^p H^k(X_t, \mathbb{C})$. Consider the subbundle $F^p\mathcal{V} \subset (\mathcal{V}, \nabla$ whose fiber is $F^p H^k(X_t, \mathbb{C})$. Notice that this subbundle is just the pullback of the tautological bundle $S \to \operatorname{Gr}$ along the period map:

$$\mathcal{P}_p^* S = F^p\mathcal{V}.$$

Since we don't know yet that the period map is holomorphic we only know that $\mathcal{P}_p^* S$ is a smooth bundle.

Theorem 361 (Griffiths)**.** *The period map $\mathcal{P}_p$ is holomorphic.*

Proof. Let $f : X \to Y$ be a smooth map of complex manifolds. Notice that f is holomorphic if and only if $df : TX_{\mathbb{C}} \to TY_{\mathbb{C}}$ sends $T^{0,1} \mapsto 0$. Thus we study

$$d\mathcal{P}_{p,t} : T_t U \to T_{F^p H^k(X_t, \mathbb{C})} \operatorname{Gr}(f_p, H^k(X, \mathbb{C})) \cong \operatorname{Hom}_{\mathbb{C}}(F^p H^k, H^k / F^p H^k).$$

Fix $u \in T_{t,u}$ and fix $s_t \in F^p H^k(X_t, \mathbb{C})$. We compute

$$d\mathcal{P}_p(u)(s_t) = d_u(\tilde{s})(t) \mod F^p H^k = \nabla_u^{\mathsf{GM}}(\tilde{s})(t) \mod F^p H^k$$

where $\tilde{s} \in \Gamma(U, \mathcal{V})$ is any section such that $\tilde{s}(t) = s_t$ and $\tilde{s}(t') \in F^p H^k(X_t, \mathbb{C})$ for all t'. The first equality follows from the copmutation at the beginning of class. But this is precisely just how we compute the Gauss-Manin connection (we choose a flat frame and just differentiate the coefficients).

Now recall that we have the Cartan-Lie formula for the Gauss-Manin connection. We can represent s_t as a closed form and put these forms together to obtain $\Omega \in G^p\mathcal{A}^k(\mathcal{X}_U)$ such that $[\Omega|_{X_t}] = \tilde{s}(t) \in F^p H^k(X_t, \mathbb{C})$. Now any take $v \in T\mathcal{X}$ such that $d\pi_* v = u$. The Cartan-Lie formula says that

$$\nabla_u(\tilde{s}) = [\iota_v(d\Omega|_{X_t})].$$

We conclude that

$$d\mathcal{P}_p(u)(s_t) = [\iota_v(d\Omega|_{X_t})] \mod F^p H^k(X_t, \mathbb{C}).$$

Now assume that v is a $(0,1)$-vector. We have that $\Omega \in F^p\mathcal{A}^k(\mathcal{X}))$ so $d\Omega \in F^p\mathcal{A}^{k+1}(\mathcal{X})$. Contracting with a vector of type $(0,1)$ also does not change p so $[\iota_v(d\Omega|_{X_t})] \in F^pH^k(X_t,\mathbb{C})$. We conclude that $d\mathcal{P}_p(u) \equiv 0 \mod F^pH^k(X_t,\mathbb{C})$. □

Now if v in the proof above were arbitrary then

$$[\iota_v(d\Omega|_{X_t})] \in F^{p-1}H^k(X_t,\mathbb{C}).$$

We thus obtain the following.

Corollary 362. *The derivative of the period map lands in:*

$$d\mathcal{P}_{p,t} : T_tU \to \mathrm{Hom}(F^pH^k, F^{p-1}H^k/F^pH^k).$$

This yields, dualizing, the famous result of Griffiths.

Corollary 363 (Griffiths transversality)**.** *The $F^p\mathcal{V}$ form a decreasing sequence of holomorphic subbundles of $\mathcal{V}$ and $\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_B$.*

This is the beginning of the theory of variations of Hodge structures. Really Griffiths transversality is just the statement that in general (even when we have singularities, etc.) the Hodge filtration is a filtration in the sense of $\mathcal{D}$-modules.

Notice that all we have is a $\mathbb{C}$-linear map $F^p\mathcal{V} \to F^{p-1}\mathcal{V} \otimes \Omega^1_B$. We now pass to the associated graded,

$$\bar{\nabla} : \mathrm{gr}^p_F \mathcal{V} \to \mathrm{gr}^{p-1}_F \mathcal{V} \otimes \Omega^1_B.$$

By doing this we gain something important: this map is $\mathcal{O}_B$-linear now: for s a section of F^p,

$$\bar{\nabla}(f\bar{s}) = \overline{\nabla(fs)} = \overline{f\nabla(s) + s \otimes df} = f\overline{\nabla(s)} = f\bar{\nabla}(\bar{s}).$$

Now this is an algebro-geometric object.

Remark 364. We have a (twisted) de Rham complex

$$\mathsf{DR}(\mathcal{V}) = \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B \to \mathcal{V} \otimes \Omega^2_B \to \cdots \to \mathcal{V} \otimes \Omega^n_B$$

since ∇ is flat. There is now a de Rham theorem

$$\mathsf{DR}(\mathcal{V}) \simeq \mathcal{L}.$$

Most of the statements in the literature about the associated graded version of this de Rham complex. From here to go back to the information about the filtrations is usually difficult homological algebra, and not much can be said in general.

53. June 4, 2018

Let $\pi : \mathcal{X} \to B$ be a family of compact Kähler manifolds. If B is contractible then there is a period map

$$\mathcal{P}_p : B \to \mathrm{Gr}(f_p, H^k(X, \mathbb{C}))$$

whose derivative we computed

$$d\mathcal{P}_{p,t} : T_tB \to \mathrm{Hom}(F^pH^k(X_t, \mathbb{C}), F^{p-1}H^k(X_t, \mathbb{C})/F^pH^k(X_t, \mathbb{C})),$$

which one can check actually descends to a map

$$d\mathcal{P}_{p,t} : T_tB \to \mathrm{Hom}(F^pH^k(X_t, \mathbb{C})/F^{p+1}H^k(X_t, \mathbb{C}), F^{p-1}H^k(X_t, \mathbb{C})/F^pH^k(X_t, \mathbb{C})) = \mathrm{Hom}(H^{p,1}X_t, H^{p-1,q+1}X_t)$$

Now since linear maps $U \to V \otimes W$ are in natural bijection with linear maps $W^* \to V \otimes U^*$. Recall that we have from last time

$$F^pH^k/F^{p+1}H^k \xrightarrow{\bar{\nabla}} F^{p-1}H^k/F^pH^k \otimes \Omega^1_{B,t}.$$

which last time we showed was $\mathcal{O}_B$-linear. Rewriting

$$d\mathcal{P}_{p,t} : T_tB \to \mathrm{Hom}(H^{p,q}X_t, H^{p-1,q+1}X_t).$$

Another natural map we might define is the Kodaira-Spencer map

$$T_tB \xrightarrow{\kappa} H^1(X_t, T_{X_t}).$$

But there is a natural map from $H^1(X_t, T_{X_t})$ to the target of the derivative of the period map. Indeed, by Dolbeault we have that

$$\mathrm{Hom}(H^{p,q}X_t, H^{p-1,q+1}X_t) \cong H^q(X, \Omega^p_{X_t})^* \otimes H^{q+1}(X_t, \Omega^{p-1}_{X_t}).$$

Moreover there is a cup product followed by contraction

$$H^1(X, T_{X_t}) \times H^q(X_t, \Omega^p_{X_t}) \to H^{q+1}(X_t, T_{X_t} \otimes \Omega^p_{X_t}) \to H^{q+1}(X_t, \Omega^{p-1}_{X_t}).$$

Remark 365. These two maps coincide, as was proved by Griffiths. This is not particularly difficult given everything we have done so far.

What are some problems that people look at when studying families of varieties?

Example 366 (Torelli problem)**.** Roughly speaking the Torelli problem is considering the injectivity of the period map $\mathcal{P}_p$: how much data about the manifold can you recover from its Hodge theoretic data? Similarly there is an infinitesimal Torelli problem, which considers the injectivity of the derivative (is it an immersion?).

The following is the simplest example that one can give.

Example 367 (Calabi-Yau manifolds)**.** We stated a serious theorem some time ago: the deformation theory of Calabi-Yau manifolds is unobstructed. Moreover, there exists locally a universal deformation space B for a fixed Calabi-Yau, call it X. This is a family $\pi : \mathcal{X} \to B$ with X the fiber of π over 0. All possible deformations are represented uniquely by definition of universal. In particular the Kodaira-Spencer map $\kappa : T_tB \simeq H^1(X_t, T_{X_t})$ is an isomorphisms an isomorphism. Suppose $n = \dim X$. Let $k = p = n$ i.e. we are interested in $F^nH^n(X, \mathbb{C}) \subset H^n(X, \mathbb{C})$. Recall $F^nH^n(X, \mathbb{C}) = H^{n,0}(X)$ which by Dolbeault is $H^0(X, \omega_X) \cong H^0(X, \mathcal{O}_X) = \mathbb{C}$. Notice that this is just one-dimensional, whence the period map reduces to

$$\mathcal{P}_n : B \to \mathrm{Gr}(1, H^n) = \mathbb{P}\, H^n(X, \mathbb{C}).$$

The derivative of the period map, by the remark above, is given

$$d\mathcal{P}_{n,t} : T_tB \xrightarrow{\sim} H^1(X_t, T_{X_t}) \to \mathrm{Hom}(F^nH^n, F^{n-1}H^n/F^nH^n).$$

Since ω_{X_t} is trivial let us fix a nowhere vanishing top form α. This form induces an isomorphism $T_{X_t} \to \Omega^{n-1}_{X_t}$. So we find

$$\mathrm{Hom}(H^0(X_t, \omega_{X_t}), H^1(X_t, \Omega^{n-1}_{X_t})) \cong \mathrm{Hom}(\mathbb{C}, H^1(X), T_{X_t})$$

and it is not hard to check that the map ψ from $H^1(X_t, T_{X_t})$ is just the identity, i.e. that $\psi(a)(\alpha) = \alpha(a)$. Hence in this case the differential of the period map is an isomorphism. In other words the infinitesimal Torelli theorem holds for Calabi-Yau manifolds.

This direction of proving Torelli-type theorems is an important industry in geometry.

Let B be a complex manifold. An **integral variation of Hodge structures** (a VHS) on B of weight k is the data of a local system $\mathcal{L}$ of free $\mathbb{Z}$-modules on B, with the associated vector bundle $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_X$ with flat connection ∇, and a decreasing filtration

$$\mathcal{V} = F^0\mathcal{V} \supset F^1\mathcal{V} \supset \cdots \supset F^k\mathcal{V} \supset F^{k+1}\mathcal{V} = 0$$

such that

$$\mathcal{V} = F^p\mathcal{V} \oplus \overline{F^{k-p+1}\mathcal{V}}$$

with conjugation with respect to $\mathcal{L}_{\mathbb{R}} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$, and such that we have Griffiths transversality:

$$\nabla(F^p\mathcal{V}) = F^{p-1}\mathcal{V} \otimes \Omega^1 B.$$

Notice that for a fixed $t \in B$ all of this data specializes to a Hodge structure of weight k. Of course, the family notion also remembers this transversality condition.

Remark 368. Of course, we could have $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ VHS's, which are weaker structures.

Remark 369. There exists a notion of polarization

$$Q : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$$

which is just a fiberwise notion of polarization, Q_t on $\mathcal{V}(t)$.

A big part of this course went into showing the following.

Example 370. If $\pi : \mathcal{X} \to B$ is a family of compact Kähler manifolds then

$$(\mathcal{L} = R^k\pi_*\mathbb{Z}, F^p\mathcal{V}, \nabla^{\mathsf{GM}})$$

is a VHS of weight k. Moreover if we have a family of projective manifolds then we obtain a polarized VHS.

For any VHS we have generalized Kodaira-Spencer maps

$$\mathrm{gr}^k_F \mathcal{V} \to \mathrm{gr}^{k-1}_F \mathcal{V} \otimes \Omega^1_B$$

whose kernels are again vector bundles that admit seminegative Kähler metrics. In other words we get inherit positivity (dualizing) from any family. For instance, one might ask whether we have a smooth family of curves of genus at least 1 over $\mathbb{P}^1$. Or over an elliptic curve? It turns out that one can use these positivity conditions coming from these VHS to show that the answer is no in both these cases. More generally one can show that certain moduli spaces are hyperbolic.

Now suppose we have a family and we know that it has singular fibres at some points. We have a VHS only away from these points, $B \setminus U$, where U is the set of points where the family is submersive. Most of the time you don't have enough technology to work with such an open manifold. Hence you try to extend a VHS. This leads to the theory of filtered $\mathcal{D}$-modules. Indeed, a VHS is the very simplest case of a $\mathcal{D}$-modules. The idea is that one should not insist on working with vector bundles. Instead one asks that we have a $\mathcal{O}_B$-module $\mathcal{M}$ with structure as

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_B.$$

The filtered condition is simply asking $F_l\mathcal{D}_X \cdot F_k\mathcal{M} \subset F_{k+l}\mathcal{M}$. This is precisely the Griffiths transversality condition! One might ask what happened to the local system side of the story. This is the story of Riemann-Hilbert correpsondence: regular holonomic $\mathcal{D}$-modules correspond to perverse sheaves.