MILNOR SEMINAR: DIFFERENTIAL FORMS AND CHERN CLASSES

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In these lectures I want to introduce the Chern-Weil approach to characteristic classes on manifolds, and in particular, the Chern classes. This approach is a bridge between algebraic topology and differential geometry constructed through the powerful machinery of differential forms. We will begin by reviewing the basics of connections and curvature before turning to characteristic forms. As one of the many applications of Chern-Weil theory, we will state and prove Chern's generalization of the Gauss-Bonnet theorem.

The fundamental geometric structure on a smooth vector bundle is a connection, which allows us to differentiate sections.

Definition 1. Let $\pi : E \to M$ be a complex vector bundle. A connection on E is a $\mathbb{C}$-linear map

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

satisfying the Leibniz rule

$$\nabla(f \otimes s) = df \otimes s + f \otimes \nabla s,$$

for $f \in C^\infty(M, \mathbb{C})$ and $s \in \Omega^0(M, E)$. We will sometimes call ∇s the covariant derivative of s.

Let's start simple – what is a connection on a trivial bundle $E = M \times \mathbb{C}^k$? Recall that a trivial bundle has a global frame, i.e. a set of sections $\{e_i\}_{i=1}^k$ such that $\{e_i(p)\}_{i=1}^k$ is a basis for the fiber $\pi^{-1}(p)$, for each $p \in M$. It follows that any section $s \in \Omega^0(M, E)$ can be written

$$s = s^i e_i$$

for $s^i \in C^\infty(M, \mathbb{C})$. Now if we are given a connection ∇ on E, we find

$$\nabla s = \nabla(s^i e_i) = ds^i \otimes e_i + s^i \nabla e_i,$$

whence we see that ∇ is completely determined by its value on the frame. Notice that ∇e_i is a one-form valued section, so we can write it again in terms of the frame

$$\nabla e_i = A_i^j e_j,$$

where $A_i^j \in \Omega^1(M, \operatorname{End} E)$ is a one-form valued endomorphism of E. We conclude, then, that any connection ∇ on a trivial bundle is determined by its connection one-form A_i^j. Conversely, given such A_i^j, we obtain a connection by defining it via the above formula. In particular, the trivial connection on $M \times \mathbb{C}^k$ is given by the choice $A_i^j = 0$ and thus annihilates the e_i.

Now suppose we have a nontrivial vector bundle $\pi : E \to M$. It is not a priori clear that connections exist.

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Proposition 2. *Connections exist.*

Proof. The vector bundle $\pi : E \to M$ is locally trivial by definition. If we denote the trivializing opens by $\{U_\alpha\}$, we find that $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^k$. On the vector bundle E_{U_α} we can construct a connection by choosing a frame $\{e_i\}$ and a connection one-form A_i^j. Carrying out this procedure on each open U_α we obtain a collection of connections ∇^α. We can now use the existence of partitions of unity to glue these connections to a connection ∇ on E. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to the trivializing cover. Then define

$$\nabla s := \sum_\alpha \chi_\alpha \nabla^\alpha \rho_\alpha(s)$$

where ρ_α denotes the restriction map. This formula does indeed yield a connection:

$$\begin{aligned}\nabla(fs) &= \sum_\alpha \chi_\alpha \nabla^\alpha \rho(fs) = \sum_\alpha \chi_\alpha \nabla^\alpha (\rho(f)\rho(s)) \\ &= \sum_\alpha \chi_\alpha \left(d(\rho(f)) \otimes \rho(s) + \rho(f)\nabla^\alpha \rho(s)\right) \\ &= df \otimes s + \sum_\alpha \rho(f)\chi_\alpha \nabla^\alpha \rho(s) \\ &= df \otimes s + f\nabla s.\end{aligned}$$

□

Now that we've established that the set of connections on a vector bundle is nonempty, it is natural to ask what structure the space of all connections carries. The first thing to notice is does not naturally carry the structure of a vector space: if ∇^1 and ∇^2 are connections on $\pi : E \to M$,

$$(\nabla^1 + \nabla^2)(fs) = 2df \otimes (\nabla^1 + \nabla^2)s + f(\nabla^1 + \nabla^2)s,$$

so their sum is not a connection. On the other hand, notice that the difference of two connections is a linear map $\Omega^0(M, E) \to \Omega^1(M, E)$ satisfying

$$(\nabla^1 - \nabla^2)(fs) = f(\nabla^1 - \nabla^2)s.$$

In other words, $\nabla^1 - \nabla^2 \in \Omega^1(M, \operatorname{End} E)$. We conclude that the set of connections on E is an affine space modelled on the infinite-dimensional vector space $\Omega^1(M, \operatorname{End} E)$.[1] In particular, we might speak of a smooth family of connections, which is just a smooth path in this affine space.[2]

Recall that linear algebraic constructions yield analagous constructions for vector bundles. We can take, for instance, the direct sum, the hom, the tensor product, the exterior product etc. of vector bundles. Given a connection on E we obtain naturally connections on these other bundles. Let's look at a few of these constructions. Consider first the direct sum: given ∇^E and ∇^F connections on bundles E and F on M, it is easy to check that

$$\nabla^{E\oplus F} := \nabla^E \oplus \nabla^F$$

[1]Recall that an affine space modelled on a vector space V is a set equipped with a transitive free action of the additive group V, i.e. a V-torsor. Intuitively, an affine structure is what is left of a vector space structure after forgetting the origin.

[2]One should exercise care about smoothness when it comes to infinite-dimensional manifolds, but I'll ignore the issue here.

is a connection on $E \oplus F$. Similarly, on the tensor product $E \otimes F$ we can define

$$\nabla^{E\otimes F} = \nabla^E \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^F$$

Next consider the bundle of algebras $\operatorname{End} E$. We want a connection

$$\nabla^{\operatorname{End} E} : \Omega^0(M, \operatorname{End} E) \to \Omega^1(M, \operatorname{End} E).$$

It is defined to be the linear map satisfying the following "product" rule. For all $s \in \Omega^0(M, E)$, we require

$$\nabla^E(As) = (\nabla^{\operatorname{End} E} A)s + A(\nabla^E s).$$

A slicker way of writing this definition is

$$\nabla^{\operatorname{End} E} = [\nabla^E, A].$$

In this notation we are considering these operators as elements of the graded Lie algebra $\operatorname{End} \Omega(M, E)$. Finally, given a smooth map $f : N \to M$ of smooth manifolds, and a vector bundle E on M with connection ∇^E, we might ask for a connection ∇^{f^*E} on the pullback bundle $f^*E \to N$. To do this, we note that $\Omega^0(N, f^*E)$ is generated as a $C^\infty(N)$-module by $f^*\Omega^0(M, E)$ (as $(f^*E)_p$ is naturally identified with $E_{f(p)}$). Hence it is enough to define

$$(f^*\nabla)(f^*s) = f^*(\nabla s)$$

and extend via the Leibniz rule.

Let us now turn to the definition of the curvature of a connection. Notice first that a connection ∇ on E extends uniquely to a degree one map

$$\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$$

defined by (and then extended linearly)

$$\nabla(\alpha \otimes s) := d\alpha \otimes s + (-1)^{|\alpha|}\alpha \otimes \nabla s$$

for $\alpha \in \Omega^\bullet(M)$ and $s \in \Omega^0(M, E)$. Hence we may as well take this to be the definition of a connection. A connection on E now determines a degree one map on the graded vector space $\Omega^\bullet(M, E)$, but it is not in general a differential. The curvature F^∇ measures precisely this failure:

$$F^\nabla := \nabla \circ \nabla = \nabla^2 = \frac{1}{2}[\nabla, \nabla] : \Omega^\bullet(M, E) \to \Omega^{\bullet+2}(M, E).$$

If we think of a connection as a first order differential operator on sections of a vector bundle, it is natural to expect that curvature be a second order differential operator. This, however, turns out not be the case.

Proposition 3. *The curvature of a connection is a two-form valued endomorphism, i.e.* $F^\nabla \in \Omega^2(M, \operatorname{End} E)$.

Proof. To check this, it is enough to show that $[F^\nabla, f] = 0$, where we view f as the multiplication-by-f endomorphism of E. We compute

$$\begin{aligned} [F^\nabla, f] &= \frac{1}{2}[[\nabla, \nabla], f] \\ &= \frac{1}{2}\left([\nabla, [\nabla, f]] + [\nabla, [\nabla, f]]\right) \\ &= [\nabla, [\nabla, f]], \end{aligned}$$

where we have used the graded Jacobi identity:

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]].$$

Notice that the inner commutator can be computed

$$[\nabla, f]s = df \wedge s + f\nabla s - f\nabla s = df \wedge s,$$

which is multiplication by df, so we conclude that

$$[F^\nabla, f] = [\nabla, df] = d^2 f = 0,$$

as desired.

For another proof, we could just compute in a local frame. If $\{e_i\}$ is a local frame and $s \in \Omega^k(M,E)$, we can write $s = s^i \otimes e_i$ where $s^i \in \Omega^k(M)$. Then, applying the Leibniz rule,

$$\begin{aligned} F^\nabla s = F^\nabla(s^i e_i) &= \nabla(ds^i \otimes e_i + (-1)^k s^i \wedge \nabla e_i) \\ &= (-1)^{k+1} ds^i \otimes \nabla e_i + (-1)^k ds^i \otimes \nabla e_i + s^i \wedge \nabla^2 e_i \\ &= s^i \wedge \nabla^2 e_i. \end{aligned}$$

Next we see that

$$\begin{aligned} \nabla^2 e_i = \nabla(A_i^j e_j) &= \nabla(dA_i^j e_j - A_i^j \nabla e_j) \\ &= dA_i^j e_j - A_i^j \wedge A_j^k e_k \\ &= (dA_i^k - A_i^j \wedge A_j^k) e_k, \end{aligned}$$

so locally the curvature acts as a two-form valued matrix, whence F^∇ is globally a two-form valued endomorphism. □

In light of this fact, one might ask what we get when we applying the connection to the curvature, ∇F, as we have a canonical connection on $\operatorname{End} E$.

Proposition 4 (Bianchi identity)**.** *The curvature F^∇ of a connection ∇ has covariant derivative zero:*

$$\nabla F^\nabla = 0.$$

Proof. Recall that the covariant derivative of an endomorphism ϕ is given by $\nabla\phi = [\nabla, \phi]$. Now we just compute:

$$\begin{aligned} \nabla F^\nabla = [\nabla, F^\nabla] &= \frac{1}{2}[\nabla,[\nabla,\nabla]] \\ &= \frac{1}{2}\left([[\nabla,\nabla],\nabla] + [\nabla,[\nabla,\nabla]]\right) \\ &= 0, \end{aligned}$$

where in the last step we have used the graded symmetry of the commutator. □

Remark 5. If the curvature of a connection ∇ vanishes then we say that ∇ is flat. In this case we obtain a chain complex:

$$\cdots \longrightarrow \Omega^{k-1}(M,E) \xrightarrow{\nabla} \Omega^k(M,E) \xrightarrow{\nabla} \Omega^{k+1}(M,E) \longrightarrow \cdots$$

which one might think of as a "twisted" de Rham complex of M.

The curvature of a connection is the key ingredient needed to construct the Chern classes.

Definition 6. Let (E, ∇) be a complex vector bundle with connection. We define the total Chern form

$$c(E, \nabla) := \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla}\right) \in \Omega^{\mathrm{even}}(M).$$

The kth Chern form $c_k(E, \nabla)$ is the homogeneous degree k component of $c(E, \nabla)$.

Let us explain precisely what we mean. In a local frame the curvature F^{∇} is the operator multiplying by a matrix of 2-forms $\Omega = dA - A \wedge A$, where A is the matrix of one-forms associated to ∇. Hence the expression inside the determinant is a matrix valued in a commutative algebra, and we may take a determinant, obtaining a differential form of even degree. To check that this form is globally defined, notice that a change of frame changes Ω to $T\Omega T^{-1}$ for some invertible $k \times k$ matrix T. The definition of $c(E, \nabla)$ is clearly invariant under such a conjugation.

The connection to the usual Chern classes starts to become apparent given the following result.

Proposition 7. *The Chern form $c(E) = c(E, \nabla)$ is closed and independent of connection.*

Proof. Let us first prove that $dc(E, \nabla) = 0$ in a local frame. We will use the identity We will use the fact that if $\phi \in \Omega^{\bullet}(M, \mathrm{End}\, E)$ then $d \operatorname{tr} \phi = \operatorname{tr} \nabla \phi$. Indeed, in the local frame $\nabla = d + A$ whence

$$\begin{aligned}\operatorname{tr} \nabla \phi &= \operatorname{tr}[\nabla, \phi] = \operatorname{tr}[d, \phi] + \operatorname{tr}[A, \phi] \\ &= \operatorname{tr}(d\phi) = d \operatorname{tr} \phi,\end{aligned}$$

where the last expression is the trace of the endomorphism given by the matrix $d\phi$ (and not the application of ϕ then d). Notice that we have used that the cyclicity of the trace. From this fact it follows immediately that for any nonnegative power r, $d \operatorname{tr} \Omega^r = \operatorname{tr} \nabla(\Omega^r) = 0$ by the Bianchi identity.

To reduce closedness to the fact just proven, let us use the following trickery, where the exponential and logarithm here are just polynomials, due to nilpotence:

$$\begin{aligned}\det\left(\mathrm{id} + \frac{1}{2\pi i}\Omega\right) &= \exp\log\det\left(\mathrm{id} + \frac{1}{2\pi i}\Omega\right) \\ &= \exp\operatorname{tr}\log\left(\mathrm{id} + \frac{1}{2\pi i}\Omega\right) \\ &= 1 - i\frac{\operatorname{tr}\Omega}{2\pi} + \frac{\operatorname{tr}\Omega^2 - (\operatorname{tr}\Omega)^2}{8\pi^2} + \cdots\end{aligned}$$

Using the Mercator series for the logarithm, we expand the inside of the trace in terms of powers of $\Omega/2\pi i$. Now hitting the equation with d, we obtain zero because the chain rule pulls out a factor that looks like d applied to a sum of traces of powers of Ω.

We will omit the proof of independence, because it is a somewhat lengthy computation that is similar to the above one. The idea is straightforward: take the smooth family of connections $\nabla_t = \nabla_0 + t\omega$. Since the space of connections is an affine space, it is enough to show that $[c(E, \nabla_0)] = [c(E, \nabla_1)]$. Indeed, one can consider $c(E, \nabla_t)$ and take its t-derivative, and one finds that the resulting expression is exact. This type of result is known as a transgression formula. □

Corollary 8. *The Chern form $c(E,\nabla)$ defines a cohomology class associated to the bundle E, independent of the choice of connection, which we denote by $c(E) \in \mathrm{H}^{even}_{dR}(M)$, and call the total Chern class of E.*

Of course, we should check that this construction satisfies the usual axiomatics of the Chern classes. These are as follows:

(i) naturality: $c(f^*E) = f^*c(E)$;
(ii) the Whitney sum formula: $c(E \oplus F) = c(E)c(F)$
(iii) normalization: our Chern class here agrees with the conventional Chern classes for line bundles.

The first two axioms are clear more or less from our definition of the Chern classes, as we will see in a moment. The last axiom takes more work, and I will omit the proof here.

The first axiom is clear from the expansion in the proof of closedness and connection-independence. For naturality, choose a connection on E and the pull-back connection on f^*E, whence

$$\begin{aligned} c(f^*E) &= \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla^{f^*E}}\right) \\ &= \det\left(\mathrm{id} + \frac{1}{2\pi i} f^* F^{\nabla}\right) \\ &= f^* \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla}\right) = f^*c(E), \end{aligned}$$

as desired. The Whitney sum formula is similarly easy: given bundles E and F on M, choose connections ∇^E and ∇^F, and let $\nabla^{E\oplus F}$ be the induced connection $\nabla^E \oplus \mathrm{id} + \mathrm{id} \oplus \nabla^F$ on the direct sum. Then since the square of a block diagonal matrix is block diagonal, we find that

$$\begin{aligned} c(E \oplus F) &= \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla^{E\oplus F}}\right) \\ &= \det\left(\mathrm{id} + \frac{1}{2\pi i}(F^{\nabla^E} \oplus \mathrm{id} + \mathrm{id} \oplus F^{\nabla^F})\right) \\ &= \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla^E}\right) \det\left(\mathrm{id} + \frac{1}{2\pi i} F^{\nabla^F}\right) \\ &= c(E)c(F), \end{aligned}$$

as desired.

Now that we have a method to construct the Chern classes of a vector bundle from differential geometric data, let us prove one of the most fundamental theorems relating the topology of a smooth manifold to its differential geometry. We will take for granted some of the properties of the Euler class. In particular, this is one of the most basic cases of the Atiyah-Singer index theorem.

Theorem 9 (Gauss-Bonnet-Chern)**.** *Let M be a smooth manifold and $E \to M$ be an oriented vector bundle with metric of even rank $2k$. Then, if ∇ is a connection on E compatible with the metric, the Euler class is represented by the Pfaffian of the curvature*

$$e(E) = \left[\frac{\mathrm{Pf}\, F^{\nabla}}{(2\pi)^k}\right].$$

Corollary 10. *If (M,g) is an compact oriented Riemannian $2n$-manifold then*

$$\chi(M) = \int e(TM) = \frac{1}{(2\pi)^n}\int_M \operatorname{Pf} F^\nabla$$

where ∇ is the Levi-Civita connection.

Let us explain the terminology used in the statements above. We will now transition to thinking about real vector bundles and I will assume familiarity with the notion of a metric g on E as well as orientability. We say that a connection ∇ is compatible with the metric if

$$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t)$$

i.e. the covariant derivative of g is zero.

Lemma 11. *If ∇ is compatible with g then the connection one-form A is locally given by a skew-symmetric matrix.*

Proof. This follows immediately from the following computation in an orthonormal local frame $\{e_i\}$:

$$\begin{aligned} 0 = dg(e_i, e_j) &= g(\nabla e_i, e_j) + g(e_i, \nabla e_j) \\ &= g(A_i^k e_k, e_j) + g(e_i, A_j^k e_k) = A_i^j + A_j^i. \end{aligned}$$

□

Next we should define the Pfaffian of an antisymmetric matrix of even dimension. We will define it for real or complex matrices, but the construction goes through for an arbitrary commutative ring (we will use it for the algebra of differential forms). In the interests of time, I will leave its basic properties as an exercise in linear algebra.

Definition 12. Let Ω be an antisymmetric matrix of even dimension k with respect to a oriented orthonormal basis $\{e_i\}$. The Pfaffian of Ω is then defined to be the scalar satisfying

$$\frac{1}{k!}\left(\sum_{i<j} \Omega_{ij} e_i \wedge e_j\right)^k = \operatorname{Pf}(\Omega) e_1 \wedge \cdots \wedge e_{2k}.$$

In particular, it is a polynomial of degree k in the matrix entries.

Example 13. Let Ω be a two-by-two matrix with a scalar x in the top-right entry. Then we see that

$$(\operatorname{Pf}\Omega) e_1 \wedge e_2 = \Omega_{12} e_1 \wedge e_2$$

whence $\operatorname{Pf}\Omega = x$. Notice that $\det \Omega = x^2 = (\operatorname{Pf}\Omega)^2$.

Lemma 14. *The Pfaffian satisfies the following properties: it changes sign under reversal of orientation but is otherwise independent of the choice of basis, it satisfies*

$$\operatorname{Pf}(T\Omega T^t) = \operatorname{Pf}(\Omega)\det T,$$

and it squares to the determinant,

$$(\operatorname{Pf}\Omega)^2 = \det \Omega.$$

With these results in hand, we can now sketch a proof of Gauss-Bonnet-Chern. Actually, we will only prove it up to a sign – fixing this ambiguity takes some more work. I should say that this is maybe not the cleanest proof of the theorem – there is a much nicer proof along similar lines due to Mathai and Quillen, which perhaps I can describe later in the quarter.

Proof of Gauss-Bonnet-Chern. Let us first check that it makes sense to consider the Pfaffian of the curvature of F^∇. Notice first that since ∇ is assumed to be compatible with the metric on E, the connection one-form matrix A is even-dimensional and antisymmetric. Now recall that F^∇ was given in coordinates by $\Omega_i^k = dA_i^k - A_i^j \wedge A_j^k$ so the curvature is evidently also even-dimensional and antisymmetric. Moreover, the Pfaffian is globally defined for the same reason the Chern form is – if we change frames, we conjugate Ω by an invertible matrix, but since our bundle is orientable and equipped with a metric, the determinant of this invertible matrix is one, and the Pfaffian does not change. Indeed, similar to our analysis for the determinant, one can check that the Pfaffian of the curvature is closed.

Let me first give a "fake proof", and then explain how to fix it. Recall that the Euler class of a complex vector bundle is equal to the top Chern class of the bundle. In our case, we have a real bundle $E \to M$ of real rank $2k$, but we may consider the top Chern class of the complexification

$$c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) = e(E \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k e(E \oplus E) = (-1)^k e(E)^2,$$

using the multiplicativity of the Euler class. Here the minus sign comes from the change in the orientation from the one induced by the complex structure to that of E. The left hand side can be written explicitly using our Chern-Weil construction of Chern classes. Fix a connection ∇ compatible with the metric on E and extend it to a connection on $E \otimes_{\mathbb{R}} \mathbb{C}$. The curvature Ω of ∇ is now an antisymmetric complex $2k \times 2k$ matrix. Next recall that the total Chern class is the characteristic polynomial, locally, for the matrix $-\Omega/2\pi i$. The top degree Chern class is thus the determinant of this matrix whence

$$\begin{aligned} c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) &= \left[\det\left(-\frac{1}{2\pi i}\Omega\right)\right] = \left[\frac{1}{(2\pi)^{2k}}(-1)^k \det \Omega\right] \\ &= \left[(-1)^k \operatorname{Pf}\left(\frac{1}{2\pi}\Omega\right)^2\right] \end{aligned}$$

It now follows by the equation above that $e(E) = \pm[\operatorname{Pf}(\Omega/2\pi)]$.

What is fake about this proof? We cannot conclude that if the squares are equal because we are not in an integral domain! Indeed, the squares are just zero for degree reasons when we are in the classical case of the tangent bundle. The elements we are considering zero-divisors because they are in fact nilpotent in the cohomology ring! To fix this, we instead carry out this computation on a large enough Grassmannian, which is the classifying space of vector bundles. The cohomology of the Grassmannian is in fact a polynomial ring in the Chern classes, whence a domain, so the argument goes through just fine.[3]

Up to sign – the determination of which takes more work – we have proved the theorem. □

[3] I should point out that I am ignoring here the issue of classifying the extra data of a connection and the metric. This probably goes through by some partition of unity argument. . .