KAN SEMINAR: BOUSFIELD'S h_*-LOCALIZATION OF SPACES

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Bousfield localization with respect to homology is, roughly speaking, a way to replace a space X by some $\tilde{X}$ such that the spaces are homology-equivalent for some generalized homology theory h_* but $\tilde{X}$ is "local". The space $\tilde{X}$ loses some homotopical information that X had, but it retains the h_*-accessible parts. The idea that such a construction might be useful is not so surprising. Recall Serre's mod $\mathcal{C}$ theory, which studies homotopy and homology up to a class $\mathcal{C}$ of groups, as well as what I talked about last time: Quillen's rational homotopy theory, in which we studied spaces only up their rational weak equivalences. Indeed, the Bousfield localization of with respect to ordinary rational homology turns out to be the rationalization of the space.

The main results of this paper are twofold. Let h_* be a generalized homology theory. Then

(a) there exists a model structure on Spaces$_*$ with weak equivalences being maps inducing isomorphisms on h_* and cofibrations as usual, with fibrations satisfying the right lifting property.
(b) for each $X \in$ hSpaces$_*$ there is an h_*-localization $X \to \tilde{X}$ such that when $h_* = H_*(-, R)$ for $R \subset \mathbb{Q}$ or $R = \mathbb{Z}_p$, $\tilde{X}$ has h_*-local homotopy.

We will not discuss the proof of (a) other than to mention that there are some set-theoretic issues that need to be dealt with.

1. Localization with respect to h_*

Let us start with some generalities on localization. Let $\mathcal{C}$ be a category with a distinguished class of morphisms $\mathcal{W} \subset \mathrm{Mor}\,\mathcal{C}$.

Definition 1. We say that an object $D \in \mathcal{C}$ is **$\mathcal{W}$-local** if for any map $w : X \to Y$ in $\mathcal{W}$ the induced map $w^* : \mathrm{Hom}(Y, D) \longrightarrow \mathrm{Hom}(X, D)$ is a bijection. In other words, D is $\mathcal{W}$-local if the functor $\mathrm{Hom}(-, D)$ sends $\mathcal{W}$ to isomorphisms in Set. A **W-localization** of an object $A \in \mathcal{C}$ is map $f : A \to D$ with D $\mathcal{W}$-local and $f \in \mathcal{W}$. Notice that if A is already $\mathcal{W}$-local then f is an isomorphism.

Example 2. The following is a useful example to keep in mind. Consider the category Ab of abelian groups with W the class of maps $A \to B$ such that $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is an isomorphism. One can check that for any abelian group D, the natural map $f : D \to D \otimes \mathbb{Q}$ is the W-localization of D.

Lemma 3. *Suppose each object of C has a W-localization. Then there is a **W-localization functor** $L : C \to C$ and a natural transformation $\eta : \mathrm{id}_C \to L$ such that $\eta_A : A \to LA$ is a W-localization for each $A \in C$. Moreover, the data of (L, η) is unique up to natural isomorphism.*

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It is not hard to see that the localization functor is idempotent, i.e. that η_{LX} is an isomorphism for all X.

Let

$$h_* : \text{hSpaces}_* \to \text{grAb}$$

be a generalized homology theory. We will denote by $\mathcal{W}$ the class of morphisms in Spaces_* inducing isomorphisms on h_* but use the terminology h_*-local instead of $\mathcal{W}$-local.

Theorem 4. *Each object of* hSpaces_* *has an* h_**-localization.*

In other words, for any space X, there exists a map $f : X \to \tilde{X}$ such that f induces isomorphisms on homology h_* and for any $g : A \to B$ inducing isomorphisms on h_* the pullback map $[B, \tilde{X}] \to [A, \tilde{X}]$ is a bijection.

Before we give and idea of how this is proved, we introduce the following useful notion.

Definition 5. We say that $\mathcal{W}$ **admits a calculus of left fractions** if:

(i) $\mathcal{W}$ is closed under (finite) compositions and contains the identities of $\mathcal{C}$;
(ii) given $X_2 \overset{w}{\leftarrow} X_1 \overset{f}{\rightarrow} X_3$ with $w \in \mathcal{W}$, there exists $X_2 \overset{g}{\rightarrow} X_4 \overset{v}{\leftarrow}$ such that $v \in \mathcal{W}$ and $vf = gw : X_1 \to X_4$;
(iii) given $X_1 \overset{w}{\rightarrow} X_2 \underset{g}{\overset{f}{\rightrightarrows}} X_3$ with $w \in \mathcal{W}$ and $fw = gw$ there exists $X_3 \overset{v}{\rightarrow} X_4$ such that $v \in \mathcal{W}$ and $vf = vg$.

In this case we obtain a characterization of local objects.

Lemma 6. *If* $\mathcal{W}$ *admits a calculus of left fractions, and* $D \in \mathcal{C}$*, then the following are equivalent:*

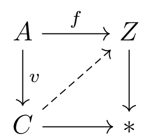
(i) D *is* $\mathcal{W}$*-local;*
(ii) each map $X \to Y$ *in* $\mathcal{W}$ *induces a surjection* $\text{Hom}(Y, D) \to \text{Hom}(X, D)$*;*
(iii) each map $D \to Y$ *in* $\mathcal{W}$ *has a left inverse.*

Proof idea. Let us suppose that the hard work has been done for us and that we know there exists a model structure on Spaces_* in which the cofibrations are the usual but the weak equivalences are h_*-weak equivalences (maps in $\mathcal{W}$). Then any map $X \to Y$ factors as $X \hookrightarrow Z \twoheadrightarrow Y$, where the first arrow is an h_*-acyclic h_*-cofibration and the second arrow is an h_*-fibration. We can in particular take $Y = *$, which yields Z an h_*-fibrant space that is h_*-weak equivalent to X.

It remains to show that Z is h_*-local. We use the fact that the class of h_*-equivalences admits a calculus of left fractions in hSpaces_* (a result observed by Adams). Then, by the above characterization of h_*-local objects, it suffices to show that the map $[B, Z] \to [A, Z]$ given by pulling back along any $w \in \mathcal{W}$ is a surjection, i.e. for each $f : A \to Z$ there exists a dashed map making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Z \\ {\scriptstyle w}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & * \end{array}$$

commute. We may assume that w is furthermore an h_*-cofibration for the following reason. There exists a space C such that we have $A \hookrightarrow C \twoheadrightarrow B$ with the first map a cofibration and the second map an acyclic fibration (in the usual sense, not an h_*-fibration!), though we won't need the fact that it is a fibration. This induces maps of sets $[B, Z] \to [C, Z] \to [A, Z]$ where the first map is a bijection since $C \twoheadrightarrow B$ is acyclic. Hence to prove that the composite is surjective it suffices to prove that the second arrow is surjective. We thus obtain a diagram



where v is an h_*-cofibration by construction (because the h_*-cofibrations are precisely the usual cofibrations) and is also in $\mathcal{W}$ because $C \to B$ and the composite $A \to B$ are. Hence we may assume that w is an h_*-acyclic h_*-cofibration whence the result follows by the model category structure described above. □

2. AN ALGEBRAIC CRITERION

Now that we know every pointed space has an h_*-localization, we ask what h_*-localization does to homotopy. The case that concerns is that of ordinary homology $H_*(-, R)$ where either $R = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ or $R = \mathbb{Z}[J^{-1}]$, which is the subring of the rationals with a set of primes J inverted.

We will take h_* and R as such for the remainder of this talk.

Theorem 7. *Let X be a connected pointed space. Then X is $H_*(-, R)$-local if and only if $\pi_n X$ is an HR-local group for $n \geqslant 1$ and $\pi_n X$ is an $H\mathbb{Z}$-local $\pi_1 X$-module for $n \geqslant 2$.*

Definition 8. Let HR be the class of maps $\alpha : G \to H$ of groups such that $\alpha_* : H_i(G, R) \to H_i(H, R)$ is an isomorphism for $i = 1$ and a surjection for $i = 2$. Here $H_i(G, R)$ is the group homology of G with coefficients in R where G acts trivially on R.

Fix a group π. Let $H\mathbb{Z}$ be the class of maps $\alpha : M \to N$ of left π-modules such that $\alpha_* : H_i(\pi, M) \to H_i(\pi, N)$ is an isomorphism for $i = 0$ and a surjection for $i = 1$.

Lemma 9. *Every group has an HR-localization and every π-module has an $H\mathbb{Z}$-localization.*

Example 10. Let G be a group and $G = \Gamma_1 G \supset \Gamma_2 G \supset \cdots$ be its lower central series. Suppose $R \otimes (\Gamma_n G/\Gamma_{n+1} G) = 0$ for some $n \geqslant 1$. Then the HR-localization of G is:

(i) for $R = \mathbb{Z}[J^{-1}]$, the map $G \to \mathbb{Z}[J^{-1}] \otimes (G/\Gamma_n G)$;
(ii) for $R = \mathbb{Z}_p$, the map $G \to \operatorname{Ext}(\mathbb{Z}_{p^\infty}, G/\Gamma_n G)$.

In particular, if G is abelian, then for the first case we obtain $G \to \mathbb{Z}[J^{-1}] \otimes G$.

Before I sketch the proof for groups I will state two lemmas that form the technical heart of the paper.

Lemma 11. *Let X be a connected pointed space and $\alpha : \pi_1 X \to G$ a map of groups. Then $\alpha \in HR$ if and only if there exists a homology equivalence $f : X \to Y$ with $f_* : \pi_1 X \to \pi_1 Y$ equal to α.*

Lemma 12. *Let X be a connected pointed space and $\alpha : \pi_n X \to M$ be a map of $\pi_1 X$-modules for $n \geqslant 2$. Then $1 \otimes \alpha : R \otimes \pi_n X \to R \otimes M$ is in $H\mathbb{Z}$ if and only if there exists a homology equivalence $f : X \to Y$ with $f_* : \pi_j X \to \pi_j Y$ an isomorphism for $j < n$ and $f_* : \pi_n X \to \pi_n Y$ equal to α.*

Proof sketch of Lemma 9. Let us sketch the proof for the case of groups. It suffices to show that if $f : X \to D$ is an $H_*(-, R)$-localization (with X connected) then $f_* : \pi_1 X \to \pi_1 D$ is an HR-localization. Applying the existence of $H_*(-, R)$-localization to the space $K(G, 1)$ proves the result.

By Lemma 11 we know that $f_* \in HR$ so it suffices to show that $\pi_1 D$ is local. Now, it turns out that HR admits a calculus of left fractions in the category of groups, whence it suffices by an earlier lemma to show that any map $w_* : \pi_1 D \to H$ in HR has a left inverse. Again by Lemma 11 there exists a homology equivalence $w : D \to Y$ for some space Y with $\pi_1 Y = H$. But because D is $H_*(-, R)$-local, w has a left-inverse which induces a left inverse for w_*.

The proof of existence of $H\mathbb{Z}$-localizations of modules is more complicated. □

Let us say that a space is **algebraically** $H_*(-, R)$**-local** if $\pi_n X$ is HR-local as a group for $n \geqslant 1$ and $\pi_N X$ is an $H\mathbb{Z}$-local $\pi_1 X$-module for $n \geqslant 2$. We wish to show that these two conditions are equivalent. We will use the following two lemmas, after which the theorem becomes straightforward.

Lemma 13. *Let X, Y be connected pointed spaces that are algebraically $H_*(-, R)$-local. If $f : X \to Y$ is a homology equivalence then f is an equivalence.*

Proof sketch. By Lemma 11, the map $f_* : \pi_1 X \to \pi_1 Y$ is in HR, so since $\pi_1 X, \pi_1 Y$ are local, f_* is an isomorphism. Now one can inductively apply Lemma 12 to show that f induces isomorphisms on higher homotopy groups. □

Lemma 14. *For each connected pointed space X there exists a homology equivalence $f : X \to Y$ such that Y is algebraically $H_*(-, R)$-local.*

Proof sketch. The group $\pi_1 X$ has an HR-localization $\pi_1 X \to G$ by Lemma 9 above. By Lemma 11 there exists a homology equivalence $f^1 : X \to Y_1$ realizing the HR-localization on fundamental groups. Similarly one obtains a homology equivalence $f^2 : Y^1 \to Y^2$ by invoking Lemma 12 on the $H\mathbb{Z}$-localization of $\pi_2 Y^1$. Repeating this for higher homotopy groups, we take Y to be the homotopy colimit of $Y^1 \to Y^2 \to Y^3 \to \cdots$, and we obtain a map $X \to Y$ with the desired properties. □

Proof of 7. Suppose X is a connected pointed space that is algebraically local. Then, since the class of homology equivalences admits a calculus of left fractions, to prove that X is $H(-, R)$-local it suffices to prove that every homology equivalence $X \to Y$ in hSpaces$_*$ has a left inverse. By Lemma 14 there exists a homology equivalence $Y \to Z$ in hSpaces$_*$ with Z algebraically local. Now the composition $X \to Y \to Z$, by Lemma 13, is a weak equivalence, which proves the claim.

Conversely, suppose X is $H(-, R)$-local. By Lemma 14 there exists a homology equivalence $X \to Y$ such that Y is algebraically local. By the previous paragraph, Y is $H(-, R)$-local, whence $X \to Y$ is a homology equivalence between $H(-, R)$-local spaces, implying that X is homotopy equivalent to Y. This proves the claim. □

In a sense, these results are quite remarkable – one can always find an h_*-local replacement for any (connected) space X, and in the case of ordinary homology (with reasonable coefficients) there is a completely algebraic detection criterion for locality. As for applications, localization turns out to be interesting in the following sense: if X_p is the $H_*(-,\mathbb{Z}/p\mathbb{Z})$-localization of X and $X_\mathbb{Q}$ is the $H_*(-,\mathbb{Q})$-localization (rationalization) of X, then

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow \\ X_\mathbb{Q} & \longrightarrow & (\prod_p X_p)_\mathbb{Q} \end{array}$$

is a homotopy pullback square. In other words, the homotopy type of X is uniquely determined by the homotopy types of its localizations. This is known as a "fracture theorem" and is originally due to Sullivan.