DIFFERENTIAL FORMS ON $B_\bullet G$

NILAY KUMAR

Abstract. We begin by recalling basic notions around differential forms on simplicial manifolds. The main object of study is the complex $A^q(B_\bullet G)$ of q-forms on the classifying space of a compact Lie group G. We compute the cohomology of this complex, following a paper of Bott's, and discuss as time permits some applications and extensions. (These notes were prepared for John Francis' seminar course in the spring quarter of 2019 at Northwestern.)

1. Simplicial manifolds

Write Δ for the simplicial indexing category and Mfld for the category of finite-dimensional smooth manifolds (without boundary).

Definition 1. A simplicial manifold $X_\bullet$ is a functor $X : \Delta^{\mathrm{op}} \to \mathsf{Mfld}$. A map of simplicial manifolds is a natural transformation of functors, and the resulting category is denoted sMfld.

Example 2. Here are a few standard examples of simplicial manifolds:

(1) First, two trivial examples. If $X_\bullet$ is zero dimensional in each degree, we obtain a simplicial set. If $X_\bullet$ is given by a fixed manifold M in each degree with the structures maps all the identity, we obtain just a smooth manifold M.

(2) Let G be a Lie group. Then we obtain $B_\bullet G$ given

$$B_n G = G^n$$

where face maps are given by multiplication (except for the first and last, which are given by projection) and degeneracy maps are given by insertion of the identity.

Similarly we obtain the universal G-bundle over $B_\bullet G$,

$$E_n G = G^{n+1}$$

whose faces maps are given by forgetting a factor and degeneracy maps are given by repeating a factor. G acts on $E_\bullet G$ diagonally on the right and we obtain a projection $E_n G \to B_n G$,

$$(h_0, \ldots, h_n) \mapsto (h_1 h_0^{-1}, \ldots, h_n h_{n-1}^{-1}).$$

(3) Given a Lie groupoid $\mathcal{G}$ we obtain a simplicial manifold by applying the usual nerve construction for groupoids. For instance $B_\bullet G$ is obtained as the nerve of the one-object Lie groupoid G and $E_\bullet G$ is obtained from the translation groupoid, which is the action groupoid for the translation action of G on itself (from the left, say).

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(4) We might think of $B_\bullet G$ as a delooping for the Lie group G. More generally, given a simplicial Lie group $G_\bullet$ there is a principal $G_\bullet$-bundle $W_\bullet G \to \bar{W}_\bullet G$ modelling the universal bundle over the delooping of $G_\bullet$.

Let me give some motivation for the introduction of simplicial manifolds. As simplicial manifolds are in particular simplicial topological spaces, it is not surprising that the novelty in studying simplicial manifolds comes from applications where the smooth structure is of interest.

Consider, for example, the Chern-Weil approach to characteristic classes. Given a complex vector bundle $\pi : E \to M$ one obtains the Chern classes of E by choosing a connection ∇ on E and then taking the trace of a certain polynomial in the curvature $F^\nabla \in \Omega^2(M)$. Changing the connection modifies the resulting form by an exact form (the differential of an expression involving the Chern-Simons secondary invariant) whence the de Rham cohomology class is independent of this choice. From a topological perspective, this cohomology class should be pulled back from an appropriate classifying space along a map classifying the vector bundle. One would like to lift this to a chain level statement but the classifying space is not a manifold in general and thus does not obviously support differential forms, let alone universal Chern-Weil forms. Such a lift to the de Rham complex was first constructed by Shulman in his Berkeley thesis, and roughly imitates the Chern-Weil construction for the simplicial G-bundle $E_\bullet G \to B_\bullet G$.

Another motivation comes from physics. In many quantum field theories the numerical invariants calculated by path integral methods fail to make sense, even at a physical level of rigor, due to the presence of what are called anomalies. Roughly speaking the presence of an anomaly signifies that the integrand is not a function but instead a section of a non-trivial line bundle. This was first made precise in an example by Quillen and then generalized by Bismut and Freed. Often the anomaly can be "trivialized" by further geometric constraints on the underlying spacetime manifold such as a spin structure. For some string theories the anomaly is trivialized by a string structure on spacetime. The string group is a 3-connected cover of the spin group but is a priori defined only as a topological group. Promoting the string group to some sort of smooth object is not so easy; if one wants to stay in the world of finite-dimensional manifolds, one is forced to view it as a 2-group: a simplicial manifold satisfying certain horn-lifting properties. Much more generally, it is interesting to ask for an analog of Lie's third theorem for L_∞-algebras. The "Lie group" corresponding to a given L_∞-algebra is naturally a simplicial Banach manifold.

Finally, as less of a motivation and more of a categorical outlook, I would like to mention that Kan simplicial manifolds offer relatively concrete and hands-on representatives for (nice enough) smooth stacks. Two Kan simplicial manifolds are to be considered equivalent as smooth stacks if they are connected by "hypercovers." These are maps of simplicial manifolds that generalize the augmentation of the Cech nerve of a good open cover. That such maps should be equivalences is already evident in the definition of a smooth manifold via atlases.

With these remarks in mind, let us turn to the notion of differential forms on simplicial manifolds.

Definition 3. Let $X_\bullet$ be a simplicial manifold. The de Rham complex of $X_\bullet$ is the cosimplicial cochain complex given as the composition

$$A^*X : \Delta \xrightarrow{X_\bullet} \mathsf{Mfld}^{\mathrm{op}} \xrightarrow{A^*(-)} \mathsf{Ch}(\mathsf{Vect}_{\mathbb{R}}).$$

For a fixed $q \geq 0$ in particular we obtain a cosimplicial vector space of q-forms on X,

$$A^qX : \Delta \to \mathsf{Vect}_{\mathbb{R}}.$$

As usual there is a functor

$$N : \mathsf{cCh}(\mathsf{Vect}_{\mathbb{R}}) \to \mathsf{Ch}(\mathsf{Ch}(\mathsf{Vect}_{\mathbb{R}}))$$

that assigns to a cosimplicial cochain complex the corresponding normalized double complex. Now to a double complex we can assign its total complex: we will often abuse terminology and refer to this complex as the de Rham complex of $X_\bullet$.

Example 4. Here are the two most basic examples:

- If $X_\bullet$ is zero-dimensional, i.e. just a simplicial set, then the de Rham complex A^*X is precisely the simplicial cochain complex of $X_\bullet$ with coefficients in $\mathbb{R}$.
- If $X_\bullet = X$ is the constant simplicial object on $X \in \mathsf{Mfld}$ then A^*X is (quasi-isomorphic to, unless one took the normalized complex) the usual complex of differential forms on X.

Remark 5. There is a product on the de Rham complex of a simplicial manifold coming from the wedge product of differential forms that turns out not to be commutative on the nose – instead, it gives the de Rham complex the structure of a C_∞-algebra. The homotopy coherent nature of the multiplication is not surprising given the first example above.

As we noted earlier, a simplicial manifold is naturally a simplicial topological space. Thus we obtain a functor

$$|-| : \mathsf{sMfld} \to \mathsf{Spaces}$$

given by first taking the underlying simplicial space and then taking the geometric realization.

Theorem 6 (Bott-Shulman-Stasheff)**.** *The simplicial de Rham complex computes the ordinary singular cohomology of the geometric realization of $X_\bullet$:*

$$H^*(A^*(X_\bullet)) \cong H^*(|X_\bullet|; \mathbb{R}).$$

2. Bott's argument

In this section we focus on q-forms on the simplicial manifold $B_\bullet G$. Recall that $A^q(B_\bullet G)$ is a cosimplicial vector space to which we assign the associated normalized complex. Following Bott, we will compute the cohomology of this complex; I thank Ezra Getzler for explaining this proof to me.

Theorem 7 (Bott '73)**.** *Let G be a compact Lie group and q be a non-negative integer. Then*

$$H^i(A^q(B_\bullet G)) \cong \begin{cases} \mathrm{Sym}^q(\mathfrak{g}^\vee)^G & i = q \\ 0 & i \neq q \end{cases}$$

The proof of this result has three ingredients:

(1) $A^q(B_\bullet G)$ can be obtained as the *basic* q-forms on the universal principal G-bundle $E_\bullet G$
(2) by compactness of G, the cosimplicial G-module $A^0(E_\bullet G)$ admits an extra (G-invariant) codegeneracy
(3) the first (nonadditive) derived functor of the exterior power functor Λ^q is a q-fold shift of the symmetric power functor Sym^q.

Before we start the proof, let me mention a result that we will use without further mention, whose proof follows by existence of averaging.

Lemma 8. *For G compact, the functor of invariants $(-)^G$ commutes with cochain cohomology for cochain complexes of G-modules (over $\mathbb{R}$).*

Let's start with the first step. Instead of working with differential forms on the classifying space $B_\bullet G$, we will work on the total space, which affords greater flexibility.

Proposition 9. *Let $\pi : P \to M$ be a smooth principal G-bundle. Then the pullback*

$$A^q(M) \xrightarrow{\pi^*} A^q(P)$$

induces an isomorphism of $A^q(M)$ onto the subspace $A^q_{basic}(P)$ of basic q-forms on P. Recall that $\omega \in A^q(P)$ is basic if it is G-invariant and horizontal:

$$R_g^*\omega = \omega \qquad \text{and} \qquad \omega|_{\pi^{-1}(m)} = 0$$

for all $m \in M$. Here $R_g : P \to P$ is the right action of $g \in G$ on P.

The proof is straightforward.

Corollary 10. *Pullback along $\pi : E_\bullet G \to B_\bullet G$ induces an isomorphism of complexes*

$$A^q(B_\bullet G) \cong A^q_{basic}(E_\bullet G) \subset A^q(E_\bullet G).$$

Definition 11. Define the cosimplicial vector space $C^\bullet\mathbb{R}$ as the composite

$$C^\bullet\mathbb{R} : \Delta \hookrightarrow \mathsf{Set} \xrightarrow{\mathsf{free}} \mathsf{Vect}_\mathbb{R}.$$

In other words, $C^n\mathbb{R}$ is the vector space of dimension $n+1$: denote by e_i the basis vector of $C^n\mathbb{R} = \mathbb{R}^{n+1}$ corresponding to $i \in [n]$. The ith coface map is the inclusion that misses e_i and the ith codegeneracy map is the surjection that sends e_i and e_{i+1} to e_i. There is a (levelwise) surjection of cosimplicial vector spaces

$$C^\bullet\mathbb{R} \xrightarrow{s} \mathbb{R} \to 0$$

to the constant one-dimensional cosimplicial vector space given (in degree n)

$$s\left(\sum_{i=0}^{n+1} a_i e_i\right) = \sum_{i=0}^{n+1} a_i.$$

If we define $\Sigma^\bullet\mathbb{R} = \ker s$ we obtain a short exact sequence of cosimplicial vector spaces

$$0 \to \Sigma^\bullet\mathbb{R} \to C^\bullet\mathbb{R} \to \mathbb{R} \to 0.$$

Remark 12. Beware that the map s is not a coaugmentation of $C^\bullet\mathbb{R}$ – it goes the wrong way!

Now for any cosimplicial vector space $V^\bullet$ we obtain cosimplicial vector spaces

$$C^\bullet V = C^\bullet \mathbb{R} \otimes_{\mathbb{R}} V^\bullet$$
$$\Sigma^\bullet V = \Sigma^\bullet \mathbb{R} \otimes_{\mathbb{R}} V^\bullet$$

that we call the cone and suspension of $V^\bullet$, respectively. They fit into a short exact sequence

$$0 \to \Sigma^\bullet V \to C^\bullet V \to V^\bullet \to 0.$$

Notice that by Eilenberg-Zilber,

$$\begin{aligned} H^* N C^\bullet V &= H^*(N(C^\bullet \mathbb{R} \otimes V^\bullet)) \\ &= H^* N C^\bullet \mathbb{R} \otimes H^* N V^\bullet \\ &= 0 \end{aligned}$$

whence

$$H^* N C^\bullet V = 0 \qquad \text{and} \qquad H^* N \Sigma^\bullet V = V[-1].$$

With this notation established, we arrive at the statement of the first result.

Lemma 13 (Decomposition lemma)**.** *Let G be a Lie group and let $\mathfrak{g}^\vee$ be the vector space of left-invariant 1-forms on G, viewed as a G-module under right-multiplication. Then*

$$A^q(B_\bullet G) \cong diag\left(A^0(E_\bullet G) \otimes \Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G.$$

In words: the cosimplicial vector space of q-forms on $B_\bullet G$ is isomorphic to the diagonal cosimplicial vector space obtained from the G-invariants of the bicosimplicial G-module in parentheses.

Proof. As noted above,

$$A^q(B_\bullet G) \cong A^q_{\text{basic}}(E_\bullet G).$$

The right hand side of the decomposition above is just another way of capturing the basic forms on $E_\bullet G$. We can see this as follows. If we choose a basis for the Lie algebra $\mathfrak{g}$ then we obtain a global frame for the cotangent bundle of G, whence $A^1(G^{n+1}) = A^0(G^{n+1}) \otimes (\mathfrak{g}^\vee)^{\oplus n+1}$. More explicitly, we have an isomorphism

$$\begin{aligned} \phi_n : A^0(E_n G) \otimes C^n \mathfrak{g}^\vee &\to A^1(E_n G) \\ \sum_{k=0}^n f_k \otimes \xi_k &\mapsto \sum_{k=0}^n f_k \cdot \pi_k^* \xi_k, \end{aligned}$$

where $\pi_k : E_n G = G^{n+1} \to G$ is projection onto the kth factor. We are interested in the subspace of horizontal forms, i.e. those that restricted to a fiber of $E_n G \to B_n G$ are zero. For $p = (p_0, \ldots, p_n) \in E_n G$ write the inclusion of a fiber

$$\begin{aligned} \iota_p : G &\to E_n G = G^{n+1} \\ g &\mapsto (p_0 g, \ldots, p_n g). \end{aligned}$$

If we now restrict along ι_p we get

$$\iota_p^* \left(\sum_{k=0}^n f_k \cdot \pi_k^* \xi_k \right) = \sum_{k=0}^n f_k \cdot (\pi_k \circ \iota_p)^* \xi_k = \sum_{k=0}^n f_k \cdot \xi_k.$$

Here we have used that the composition $\pi_k \circ \iota_p : G \to G$ is left-multiplication by p_k,

$$\pi_k(\iota_p(g)) = \pi_k(p_0 g, \ldots, p_n g) = p_k g,$$

together with the left-invariance of $\xi_k \in \mathfrak{g}^\vee \subset A^1(G)$. We thus conclude that

$$A^1_{\mathrm{horiz}}(E_n G) = A^0(E_n G) \otimes \Sigma^n \mathfrak{g}^\vee,$$

from which we obtain

$$A^q_{\mathrm{horiz}}(E_n G) = A^0(E_n G) \otimes \Lambda^q \Sigma^n \mathfrak{g}^\vee,$$

Taking invariants (and checking cosimplicial structure maps) results in the desired decomposition

$$A^q_{\mathrm{basic}}(E_\bullet G) = \mathrm{diag}\left(A^0(E_\bullet G) \otimes \Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G.$$

□

As one might expect, we can throw away the first term due to the contractibility of the universal bundle.

Lemma 14. *For G compact there is a weak equivalence*

$$\mathit{diag}\left(A^0(E_\bullet G) \otimes \Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G \simeq \left(\Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G.$$

of cosimplicial vector spaces (i.e. a quasi-isomorphism on complexes).

Proof sketch. By the (Dold and Puppe generalization of the) Eilenberg-Zilber theorem, the cohomology of the diagonal can be computed instead as the cohomology of the total complex of the Moore double complex of the given bisimplicial G-module. Notice that the cosimplicial vector space $A^0(E_\bullet G)$ admits an extra codegeneracy (as $E_\bullet G$ admits an extra degeneracy). This is not enough to prove acyclicity of the rows, however, due to the $(-)^G$. Instead we need an extra codegeneracy of cosimplicial G-modules. This is where the compactness of G comes into play: usually the extra degeneracy on $E_\bullet G$ is given by insertion of the identity as the first factor, but we can instead take an average over the insertion of every element of G, using the Haar measure. □

We now have an equivalence of cosimplicial vector spaces

$$A^q(B_\bullet G) \simeq \left(\Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G.$$

To finish the proof of Bott's result we must now compute the cohomology of the right-hand side.

Proposition 15. *There is a weak equivalence*

$$N\left(\Lambda^q \Sigma^\bullet \mathfrak{g}^\vee\right)^G \simeq \mathrm{Sym}^q(\mathfrak{g}^\vee)^G[-q].$$

The statement involves an interplay between the alternating and symmetric power functors so it is natural to start by thinking about the tensor power functor. Write S_q for the symmetric group on q letters and

$$T^q : \mathsf{Mod}_G \to \mathsf{Bimod}_{(G,S_q)}$$

for the functor sending a G-module V to the qth tensor power $V \otimes \cdots \otimes V$ equipped with the diagonal action of G and the permutation action of S_q.

Proposition 16. *There is an isomorphism*

$$H^* N(T^q \Sigma^\bullet V)^G \cong (\bar{T}^q V)^G[-q].$$

Here $\bar{T}^q V$ is $T^q V$ with $\mathbb{R}S_q$-module structure twisted by the determinant (or sign) character.

Proof. Applying Eilenberg-Zilber,

$$\begin{aligned} H^*N(T^q\Sigma^\bullet V)^G &= H^*N(\Sigma^\bullet V \otimes \cdots \otimes \Sigma^\bullet V)^G \\ &\cong H^*\left(N\Sigma^\bullet V \otimes \cdots \otimes N\Sigma^\bullet V\right)^G \\ &\cong \left(H^*N\Sigma^\bullet V \otimes \cdots \otimes H^*N\Sigma^\bullet V\right)^G \\ &\cong (\bar{T}^q V)^G[-q] \end{aligned}$$

The $\bar{T}^q$ appears instead of the untwisted T^q because the identification $N(C^\bullet \otimes D^\bullet) \simeq NC^\bullet \otimes ND^\bullet$ introduces Koszul signs upon permutation of factors. □

We are interested in subfunctors of the tensor power functor — in particular the alternating and symmetric power functors. These can be obtained by projection using central idempotents in the group algebra $\mathbb{R}S_q$.

Proof of proposition. Let $e \in \mathbb{R}S_q$ be a central idempotent,[1] and consider the corresponding direct sum decomposition

$$e \cdot T^q V \underset{e}{\overset{\iota}{\rightleftarrows}} T^q V$$

We obtain an induced decomposition

$$H^*(N\Sigma^\bullet e \cdot T^q V)^G \rightleftarrows H^*(N\Sigma^\bullet T^q V)^G \cong (\bar{T}^q V)^G[-q]$$

whence

$$H^*(N\Sigma^\bullet e \cdot T^q V)^G = e \cdot (\bar{T}^q V)^G[-q].$$

The group algebra $\mathbb{R}S_q$ admits an involution $\overline{(\cdot)}$ given by $\sigma \mapsto \bar{\sigma} = \det \sigma \cdot \sigma$. Twisting the action of S_q on $T^q V$ by this involution clearly yields $\bar{T}^q V$, whence we obtain an isomorphism of underlying vector spaces

$$(e \cdot \bar{T}^q V)^G = (\bar{e} \cdot T^q V)^G.$$

We conclude that

$$H^*(N\Sigma^\bullet e \cdot T^q V)^G \cong (\bar{e} \cdot T^q V)^G[-q].$$

Using the fact that this involution interchanges alternation and symmetrization and taking $V = \mathfrak{g}^\vee$, we obtain the proposition as well as Bott's theorem. □

3. Applications

The computation of differential forms on $B_\bullet G$ has a number of interesting applications; here I will focus on some immediate corollaries pertaining to the topology/geometry of the classifying stack itself.

There is a relatively recent notion of (shifted) symplectic structures on (derived) stacks due to Pantev-Toen-Vaquié-Vezzosi: one of their fundamental examples is the classifying stack of a reductive algebraic group. In the smooth setting that we are working in, we obtain the following.

[1] E.g. if $q = 2$ then $e = (1+\tau)/2$ or $e = (1-\tau)/2$, where τ is the transposition, in the case of symmetric and alternating powers respectively.

Corollary 17. *For G a compact Lie group, the set of 2-shifted symplectic structures on $B_\bullet G$ is the set of Killing forms on $\mathfrak{g}$ (nondegenerate invariant symmetric bilinear forms on $\mathfrak{g}$).*

Once we define what a 2-shifted closed form on a simplicial manifold is, this corollary is immediate. That the notion needs defining in the first place stems from the fact a cocycle in a complex is not a homotopy invariant object. The way out is to notice that the de Rham complex of manifold comes with a canonical filtration known as the Hodge filtration

$$F^pA^*X = 0 \to A^pX \to A^{p+1}X \to \cdots ,$$

and that

$$H^0(F^pA^*X[p]) = A^p_{\mathrm{cl}}X.$$

Definition 18. Let $X_\bullet$ be a simplicial manifold. The complex of closed p-forms on $X_\bullet$ is defined to be the total complex of the double complex obtained by applying F^pA^* to $X_\bullet$,

$$A^p_{\mathrm{cl}}(X_\bullet) = F^pA^*X_\bullet[p].$$

An n-shifted closed p-form ω is a degree n cocycle of this complex:

$$\omega \in Z^{n+p}(F^pA^*X_\bullet).$$

Let's unpack what an n-shifted closed p-form is, more concretely. We have

$$\omega = (\omega_0, \ldots, \omega_n), \qquad \omega_i \in A^{p+n-i}(X_i)$$

such that

$$\begin{aligned} \delta\omega_n &= 0 \\ \delta\omega_{n-1} &= \pm d\omega_n \\ &\vdots \\ \delta\omega_0 &= \pm d\omega_1 \\ d\omega_0 &= 0. \end{aligned}$$

In other words, a n-shifted closed p-form is a δ-closed p-form ω_n on X_n (hence the n-shifted p-form) whose de Rham differential is δ-exact via $\omega_{n-1} \in A^{p+1}(X_{n-1})$. Similarly ω_{n-1} is closed up to a δ-exact form, and so on and so forth until ω_0, which is a closed $(p+n)$-form on X_0.

Example 19. For $X_\bullet = B_\bullet G$ in particular we have

$$A^p_{\mathrm{cl}}(B_\bullet G) = 0 \to A^p(G) \to A^p(G^2) \oplus A^{p+1}G \to A^p(G^3) \oplus A^{p+1}(G^2) \oplus A^{p+2}G \to \cdots$$

A 2-shifted closed 2-form on $B_\bullet G$ is thus the data of

$$\omega_1 \in A^3(G) \qquad \text{and} \qquad \omega_2 \in A^2(G^2)$$

such that

$$d\omega_1 = 0 \qquad \delta\omega_1 = \pm d\omega_2 \qquad \delta\omega_2 = 0.$$

Proposition 20. *The cohomology of the complex of closed p-forms on the simplicial manifold $B_\bullet G$ is*

$$H^q(A^p_{cl}B_\bullet G) = \begin{cases} 0 & q < p \\ \mathrm{Sym}^q(\mathfrak{g}^\vee)^G & q \geq p \end{cases}$$

Proof. Consider the spectral sequence for the double complex of closed p-forms on $B_\bullet G$ where on the E_2-page we first take horizontal cohomology and then vertical cohomology. Bott's computation tells us that the qth row is $\mathrm{Sym}^q(\mathfrak{g}^\vee)^G$ concentrated in degree q. Thus there are no nontrivial differentials and the spectral sequence collapses at the E_2 page. The result follows. □

We see in particular that the set of 2-shifted closed 2-forms on $B_\bullet G$ is in bijection with G-invariant symmetric bilinear forms on $\mathfrak{g}$. Now, I don't want to go into the notion of nondegeneracy of a 2-form on a simplicial manifold, but it is hopefully reasonable that it restricts us to nondegenerate such bilinear forms, so I will declare victory here.

Having symplectic structures on classifying stacks is useful, at least theoretically, because moduli space are often constructed as mapping stacks with target classifying stacks. By arguments that are known as AKSZ-type arguments, symplectic structures can be "transgressed" from the classifying stack to the moduli spsace (at the cost of further shifting). This provides a conceptual explanation for the presence of symplectic (or sometimes more generally Poisson) structures on moduli spaces.

Remark 21. Let G be a matrix group like $U(n)$. In this case the data of a symplectic structure is the data of forms ω_1 and ω_2 living on products of $U(n)$ and so it is reasonable to hope that, given a Killing form on $\mathfrak{u}(n)$, there is some sort of concrete formula for these differential forms. This is the due to Shulman, who proceeds, roughly, by choosing a universal connection on $E_nG \to B_nG$ and using the Killing form on $\mathfrak{u}(n)$ to construct ω_1 and ω_2 following Chern and Weil. In particular, the symplectic form on $B_\bullet U(n)$ is constructed as the Shulman representative for the second Chern class c_2.

connectedness?

Remark 22. We have focused here on $B_\bullet G$, a simplicial representative for the classifying space of principal G-bundles. There exist higher categorical analogs of principal bundles known as gerbes: instead of G-valued transition functions on double overlaps of a cover, gerbes are defined by transition functions on triple overlaps. At least in the abelian case, there is a nice theory of these "bundle gerbes" and they are classified by the higher delooping B^2A (or if you wish $\bar{W}^2A$).

It should be straightforward to prove an analog of Bott's result for the classifying spaces B^nA for A a compact abelian Lie group ($A = U(1)$ is of interest, say):

$$H^i(A^q(B^n_\bullet A)) \cong \begin{cases} \mathrm{Sym}^q(\mathfrak{a}^\vee)^A & q = 2n \\ 0 & q \neq 2n \end{cases}$$

and to conclude that B^nA is equipped with a $2n$-shifted symplectic structure, though I haven't worked out the details (I expect the proof to be more or less identical if one uses the model $\bar{W}^nA$).